AB Calculus – Step-by-Step
1. Limits

Consider the functions \( f(x) = \frac{ax^2 - b}{x^2 - 4} \) and \( g(x) = \frac{x^2 - cx + d}{x^2 - 4} \).

a. If \( \lim_{x \to 3} f(x) = 6 \), find a relationship between \( a \) and \( b \). (1)

\[
\lim_{x \to 3} f(x) = \frac{9a - b}{9 - 4} = 6 \Rightarrow 9a - b = 30 \quad \text{or} \quad b = 9a - 30
\]

1 pt for relationship

b. If, in addition, \( \lim_{x \to 1} f(x) = 1 \), find the values of \( a \) and \( b \). (3)

\[
\lim_{x \to 1} f(x) = \frac{a - b}{1 - 4} = 1 \Rightarrow a - b = -3 \quad \text{or} \quad b = a + 3
\]

\[
a + 3 = 9a - 30 \Rightarrow 8a = 33 \Rightarrow a = \frac{33}{8}
\]

\[
b = a + 3 = \frac{57}{8}
\]

1 pt for \( b = a + 3 \)
1 pt for \( a \)
1 pt for \( b \)

c. Find \( \lim_{x \to -\infty} f(x) \). (1)

\[
\lim_{x \to -\infty} f(x) = a = \frac{33}{8}
\]

1 pt for answer

d. If \( \lim_{x \to 2} g(x) = 3 \), find \( c \) and \( d \). (4)

\[
\lim_{x \to 2} g(x) = \frac{x^2 - cx + d}{(x-2)(x+2)} = 3
\]

For this to happen, \( x^2 - cx + d \) must factor into \((x-2)(x-k)\)

\[
\lim_{x \to 2} g(x) = \frac{(x-2)(x-k)}{(x-2)(x+2)} = \frac{x-k}{x+2} = \frac{2-k}{4} = 3
\]

\[
2 - k = 12 \Rightarrow k = -10
\]

\[
x^2 - cx + d = (x-2)(x+10) = x^2 + 8x - 20
\]

\[
c = -8, d = -20
\]

1 pt for realizing numerator must factor into factor into \((x-2)(x-k)\)
1 pt for \( x^2 + 8x - 20 \)
1 pt for \( c \), 1 pt for \( d \)
AB Calculus – Step-by-Step
2. Average/Instantaneous Rate of Change

The Newton Bridge is a toll bridge that gets a lot of traffic during the two hour rush hour period starting at 4:00 PM. The total number of vehicles that go through the northbound is a differentiable function \( V \) of time \( t \). A table of selected values of \( V \) is given for the time interval \( 0 \leq t \leq 2 \) where \( t = 0 \) corresponds to 4 PM.

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V(t) ) (vehicles)</td>
<td>0</td>
<td>200</td>
<td>425</td>
<td>755</td>
<td>unknown</td>
</tr>
</tbody>
</table>

The southbound tollbooth also monitors vehicles that pass through it by estimating the rate of vehicles that come through. This rate is given by the differentiable function \( R \) of \( t \). A table of selected values of \( R \) is given for the time interval \( 0 \leq t \leq 2 \) where \( t = 0 \) corresponds to 4 PM.

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(t) ) (vehicles per hour)</td>
<td>390</td>
<td>425</td>
<td>425</td>
<td>535</td>
<td>460</td>
</tr>
</tbody>
</table>

a. Approximate \( V'(0.75) \). Show the computation that leads to your answer. Explain the meaning of your answer in context to the problem situation using correct units of measure.  

\[
V'(0.75) = \frac{V(1) - V(0.5)}{1 - 0.5} = \frac{425 - 200}{0.5} = 450
\]

Vehicles are passing through the northbound tollbooth at 4:45 PM at approximately 450 vehicles/hour.

b. Approximate \( R'(1.75) \). Show the computation that leads to your answer. Explain the meaning of your answer in context to the problem situation using correct units of measure.

\[
R'(1.75) \approx \frac{R(2) - R(1.5)}{2 - 1.5} = \frac{460 - 535}{0.5} = -150
\]
The rate of vehicles that are passing through the southbound tollbooth at 5:45 PM is decreasing at approximately 150 vehicles per hour per hour.

c. The number of vehicles that have passed through the northbound tollbooth by 6 PM is unknown. It is known that the average rate of change of vehicles passing through this booth between 4 PM and 6 PM is the same as the average rate of change of vehicles passing through this booth between 5 PM and 6 PM. Approximate the total vehicles having passed through the northbound tollbooth by 6:00 PM. Show the computation that leads to your answer.

\[
\frac{V(2) - V(0)}{2 - 0} = \frac{V(2) - V(1)}{2 - 1} \Rightarrow V(2) = \frac{V(2) - V(1)}{1} = \frac{V(2) - 425}{2} = \frac{V(2) - 425}{1} = 850
\]

\[
2V(2) - 850 = V(2) \Rightarrow V(2) = 850
\]
d. (1) 1 pt for context for parts a and b (4:45 PM, 5:45 PM)

© 2012 www.mastermathmentor.com

Illegal to post this document on the Internet
AB Calculus – Step-by-Step
3. Basic Derivatives and Limits

For the following problems, \( f(x) = \frac{2}{x^2} \) and \( g(x) = x^2 - 6 \).

a. Find \( \lim_{x \to -\infty} f(x)g(x) \). \( \text{(1)} \)

\[
\lim_{x \to -\infty} f(x)g(x) = \lim_{x \to -\infty} \frac{2(x^2 - 6)}{x^2} = 2
\]

1 pt answer

b. Find \( \frac{d}{dx} [f(x)g(x)] \). \( \text{(1)} \)

\[
\frac{d}{dx} \left[ \frac{2x^2 - 12}{x^2} \right] = \frac{d}{dx} \left( \frac{2 - 12}{x^2} \right) = \frac{12(2x)}{x^4} = \frac{24}{x^3}
\]

1 pt answer

c. Find \( \frac{d}{dx} [x \cdot g(f(x))] \). \( \text{(2)} \)

\[
x \cdot g(f(x)) = x \left( \frac{2}{x^2} \right)^2 - 6 = \frac{4}{x^3} - 6x
\]

\[
\frac{d}{dx} \left( \frac{4}{x^3} - 6x \right) = \frac{-4(3x^2)}{x^6} - 6 = -\frac{12}{x^4} - 6
\]

1 pt for finding \( x \cdot g(f(x)) \)
1 pt answer

d. If \( \frac{1}{y} = f(x) + 1 \), find \( \frac{dy}{dx} \). \( \text{(2)} \)

\[
y = \frac{1}{f(x) + 1} = \frac{1}{\frac{2}{x^2} + 1} = \frac{x^2}{x^2 + 2}
\]

\[
\frac{dy}{dx} = \frac{(x^2 + 2)(2x) - x^2(2x)}{(x^2 + 2)^2} = \frac{4x}{(x^2 + 2)^2}
\]

1 pt for \( y = \frac{x^2}{x^2 + 2} \)
1 pt for \( \frac{dy}{dx} \)

e. Find \( \lim_{\Delta x \to 0} \frac{f'(-2 + \Delta x) - f'(-2)}{\Delta x} \). \( \text{(3)} \)

This is asking for the derivative of \( f'(x) \) or \( f''(x) \) at \( x = -2 \)

\[
f'(x) = \frac{-4}{x^3} \text{ so } f''(x) = \frac{4(3x^2)}{x^6} = \frac{12}{x^3}
\]

\[
f''(-2) = \frac{12}{16} = \frac{3}{4}
\]

1 pt for realizing is \( f''(x) \) is needed
1 pt for \( f''(x) \)
1 pt for \( f''(-2) \)
The functions $f$ and $g$ are differentiable for all real numbers $g$. The table above gives values of the function and their first derivatives at selected values of $x$.

a. If the function $h$ is given by $h(x) = \frac{f(x)}{g(x)} + x$, find $h'(1)$.  
\[ h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} + 1 \]
\[ h'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} + 1 = \frac{5(-3) - 4(2)}{5^2} + 1 = \frac{-23}{25} + 1 = \frac{2}{25} \]

b. If the function $r$ is given by $r(x) = -2f(x)g(x)$, find the equation of the tangent line to $r(x)$ at $x = 2$.  
\[ r(2) = -2f(2)g(2) = -2(-3)(4) = 24 \]
\[ r'(x) = -2\left[f(x)g'(x) + g(x)f'(x)\right] \]
\[ r'(2) = -2\left[f(2)g'(2) + g(2)f'(2)\right] = -2\left[(-3)(6) + 4(-1)\right] = 44 \]
\[ y - 24 = 44(x - 2) \text{ or } y = 44x - 64 \]

(c) If the function $v$ is given by $v(x) = \frac{f(x) - 1}{f(x)}$, find the slope of the line normal to $v$ at $x = 3$.  
\[ v(x) = \frac{f(x) - 1}{f(x)} = 1 - \frac{1}{f(x)} \]
\[ v'(x) = \frac{f(x)(0) - (-1)f'(x)}{f(x)^2} = v'(3) = \frac{f'(3)}{f(3)^2} = \frac{8}{\pi^2} \]
\[ \text{slope of normal line is } \frac{-\pi^2}{8} \]

(d) If the function $w$ is given by $w(x) = xf(x)$ and $w'(4) = 9$, find $f'(4)$.  
\[ w'(x) = xf'(x) + f(x) \]
\[ w'(4) = 4f'(4) + f(4) = 9 \Rightarrow 4f'(4) - 5 = 9 \]
\[ 4f'(4) = 14 \Rightarrow f'(4) = \frac{7}{2} \]
Let \( f(x) = x + \sin x \) which is defined on \([0, 2\pi]\).

a. Find all exact values of \( x \) for which \( f'(x) = 1.5 \).  

\[
f'(x) = 1 + \cos x = 1.5
\]
\[
\cos x = 0.5 \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}
\]

2 pts for answers

b. If \( g(x) = \frac{f(x)}{x} \), find the equation to the tangent line to \( g \) at \( x = \frac{\pi}{2} \).  

\[
g(x) = \frac{x + \sin x}{x} = 1 + \frac{\sin x}{x}
\]
\[
g\left(\frac{\pi}{2}\right) = 1 + \frac{1}{\frac{\pi}{2}} = 1 + \frac{2}{\pi}
\]
\[
g'(x) = \frac{xcosx - \sin x}{x^2} \Rightarrow g'\left(\frac{\pi}{2}\right) = \frac{\frac{\pi}{2} \cdot 0 - 1}{\left(\frac{\pi}{2}\right)^2} = -\frac{4}{\pi^2}
\]
\[
y - \left(1 + \frac{2}{\pi}\right) = -\frac{4}{\pi^2} \left(x - \frac{\pi}{2}\right)
\]

1 pt for \( g\left(\frac{\pi}{2}\right) \)

1 pt for \( g'\left(\frac{\pi}{2}\right) \)

1 pt for equation

c. If \( h(x) = \csc x \), find all values of \( x \) on \([0, 2\pi]\) where \( f'(x) = h'(x) \).  

\[
h(x) = \frac{1}{\sin x} \Rightarrow h'(x) = -\frac{\cos x}{\sin^2 x}
\]
\[
1 + \cos x = -\frac{\cos x}{\sin^2 x}
\]
\[
\sin^2 x (1 + \cos x) = -\cos x \Rightarrow \sin^2 x (1 + \cos x) + \cos x = 0
\]
\[
x = 2.032, x = 4.251
\]

1 pt for setting student’s \( f'(x) = h'(x) \)

2 pt for answers
AB Calculus – Step-by-Step
6. Linear Approximation

A line is tangent to the graph of \( f(x) = 25 - x^2 \) at point \( P \), as shown in the figure above.

a. Show that the \( x \)-coordinate of point \( P \) is 3. Explain your reasoning.  

\[
\text{Point } P(3,16) \\
f'(x) = m_{\text{parabola}} = -2x \\
\text{At } x = 3, \text{ the slope of the tangent line to the parabola } = \text{slope of the line} \\
m = f'(3) = -6 \\
y - 34 = -6x \quad \text{or} \quad y = 34 - 6x
\]

b. Find the equation of the line.  

\[
m = -2(3) = -6 \\
y - 34 = -6x \quad \text{or} \quad y = 34 - 6x
\]

c. Show that the difference between \( f(3 + a) \) and the linear approximation to \( f(x) \) at \( x = 3 + a \) where \( a \) is a constant gives the same value as the difference between \( f(3 - a) \) and the linear approximation to \( f(x) \) at \( x = 3 - a \).

\[
\begin{align*}
\text{At } x = 3 + a & \quad \text{At } x = 3 - a \\
\text{Approximation } = 34 - 6(3 + a) = 16 - 6a & \quad \text{Approximation } = 34 - 6(3 - a) = 16 + 6a \\
f(3 + a) = 25 - (3 + a)^2 = 16 - 6a - a^2 & \quad f(3 - a) = 25 - (3 - a)^2 = 16 + 6a - a^2 \\
\text{Difference } = 16 - 6a - (16 - 6a - a^2) = a^2 & \quad \text{Difference } = 16 + 6a - (16 + 6a - a^2) = a^2
\end{align*}
\]

1 pt for approximation at \( 3 + a \)  
1 pt for \( f(3 + a) \)  
1 pt for approximation at \( 3 - a \)  
1 pt for \( f(3 - a) \)  
1 pt for showing differences are the same

© 2012 www.mastermathmentor.com

Illegal to post this document on the Internet
AB Calculus – Step-by-Step
7. Chain Rule/Trig

<table>
<thead>
<tr>
<th>x</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{3\pi}{4}$</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>$\frac{1}{a}$</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>-6</td>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$2a^2$</td>
<td>9</td>
<td>3</td>
</tr>
</tbody>
</table>

The functions $f'$ and $g$ are differentiable for all real numbers $x$. The table above gives values of the function and their first derivatives at selected values of $x$ with $a$ being a constant.

a. If $h(x) = \sin\left(f\left(\frac{3\pi}{4}\right)\right) = \frac{\sqrt{2}}{2}$, write an equation of the line tangent to $h$ at the point where $x = 1$. (3)

$h(1) = \sin f(1) = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$h'(x) = \cos(f(x)) \cdot f'(x) \Rightarrow h'(1) = \cos(f(1)) \cdot f'(1) = \cos\left(\frac{3\pi}{4}\right) \cdot \sqrt{2} = \frac{-\sqrt{2}}{2} \cdot \sqrt{2} = -1$

$y - \frac{\sqrt{2}}{2} = -1(x - 1)$ or $y = -x + 1 + \frac{\sqrt{2}}{2}$

1 pt for $h(1)$
1 pt for $h'(1)$
1 pt for equation

b. If $r(x) = \frac{1}{\sqrt{g(2x)}}$, find $r'(x)$ at $x = 2$. (2)

$r(x) = \left[g(2x)\right]^{-\frac{1}{2}} \Rightarrow r'(x) = \left[-\frac{1}{2} \cdot g(2x)^{-\frac{3}{2}} \cdot g'(2x) \cdot 2 = -\frac{g'(2x)}{g(2x)^{\frac{3}{2}}} \right]$ $\frac{r'(2)}{g(4)^{\frac{3}{2}}} = \frac{-3}{9^{\frac{3}{2}}} = \frac{-3}{27} = \frac{-1}{9}$

1 pt for $r'(x)$
1 pt for $r'(2)$

c. Find the value(s) of $a$ if the tangent lines to $f(g(x))$ and $g(f(x))$ are perpendicular at $x = 3$. (4)

$\left[f'(g(x)) \cdot g'(x)\right] \cdot \left[g'(f(x)) \cdot f'(x)\right] = -1$ at $x = 3$

$[f'(g(3)) \cdot g'(3)] \cdot [g'(f(3)) \cdot f'(3)] = -1$

$[f'(4) \cdot \frac{1}{2}] \cdot [g'(1) \cdot 5] = -1$

$2a^2 \left(\frac{1}{2}\right) \left(\frac{1}{a}\right)(5) = -1$

$5a = -1 \Rightarrow a = \frac{-1}{5}$

1 pt for derivative of $f(g(x))$ at $x = 3$
1 pt for derivative of $g(f(x))$ at $x = 3$
1 pt for $(m_1)(m_2) = -1$ or $m_2 = \frac{-1}{m_1}$
1 pt for answer
Parts a, b, and c all refer to \( f(x) \), given by \( f(x) = x^2 - x - 6 \) which is defined on \([0, 6]\).

a. Write an equation of the line tangent to \( f \) at the point where \( x = 4 \).  

\[
\begin{align*}
\quad f(4) &= 4^2 - 4 - 6 = 6 \\
\quad f'(x) &= 2x - 1 \\nto f'(4) &= 2(4) - 1 = 7 \\
y - 6 &= 7(x - 4) \quad \text{or} \quad y = 7x - 22
\end{align*}
\]

1 pt for \( f(4) \)
1 pt for \( f'(x) \)
1 pts for equation

b. If \( g(x) = [f(x)]^2 \), write an equation of any horizontal tangent lines to \( g \). Show how you arrive at your answer.  

\[
\begin{align*}
g'(x) &= 2f(x)f'(x) = 2(x^2 - x - 6)(2x - 1) = 2(x - 3)(x + 2)(2x - 1) = 0 \\
x &= 3, -2, \frac{1}{2} \quad \text{On domain, } x = 3, \frac{1}{2} \\
g(3) &= 0 \implies y = 0 \quad g\left(\frac{1}{2}\right) = \left(\frac{1}{4} - 1 - 6\right)^2 = \left(-\frac{25}{4}\right)^2 \implies y = \frac{625}{16}
\end{align*}
\]

1 pt for \( g'(x) \)
2 pts for equation
\((-1)\) for extraneous equations

c. If \( h(x) = \frac{1}{f(2x)} \), find all values of \( x \) where the tangent lines to \( h \) are either horizontal or do not exist on the interval \([0, 6]\). Show how you arrive at your answer.  

\[
\begin{align*}
h(x) &= \left[f(2x)\right]^{-1} = \frac{1}{4x^2 - 2x - 6} \\
h'(x) &= \frac{-8x - 2}{(4x^2 - 2x - 6)^2} = \frac{-2(4x - 1)}{\left[2(2x^2 - x - 3)\right]^2} = \frac{-2(4x - 1)}{\left[(2x - 3)(x + 1)\right]^2}
\end{align*}
\]

On \([0,6]\), horizontal tangent line at \( x = \frac{1}{4} \)
and tangent line doesn’t exist at \( x = \frac{3}{2} \).

1 pt for \( h'(x) \)
1 pt for horizontal
1 pt for no tangent line
\((-1)\) for extraneous values
Let \( f(x) = \frac{e + \ln x}{x^2} \).

a. Find the average rate of change of \( f \) from \( x = 1 \) to \( x = e \). (2)

\[
\frac{f(e) - f(1)}{e - 1} = \frac{e + \ln e - e + \ln 1}{e^2 - 1^2} = \frac{e + 1}{e - 1} = \frac{e + e^3}{e^2 - e} \text{ or } \frac{e + e^3}{e^3 - e^2} \]

1 pt for average rate of change formula
1 pt for answer

b. Write an equation of the line tangent to \( f \) at \( x = 1 \). (3)

\[
f'(x) = \frac{x^2 \left( \frac{1}{x} \right) - 2x(e + \ln x)}{x^4} = x \left[ 1 - 2(e + \ln x) \right] \]
\[
f'(x) = \frac{1 - 2(e + \ln x)}{x^3} \Rightarrow f'(1) = \frac{1 - 2(e + \ln 1)}{1^3} = 1 - 2e \]
\[
f(1) = \frac{e + \ln 1}{1^2} = e \]
Tangent line: \( y - e = (1 - 2e)(x - 1) \)

1 pt for \( f'(x) \)
1 pt for \( f(1) \)
1 pt for answer

c. Find the \( x \)-coordinate of the point on \( f \) at which the tangent line to \( f \) is horizontal. (2)

\[
f'(x) = \frac{1 - 2(e + \ln x)}{x^3} = 0 \]
\[
2 \ln x = 1 - 2e \Rightarrow \ln x = \frac{1 - 2e}{2} \]
\[
x = \frac{1 - 2e}{2} \]

1 pt for \( f'(x) = 0 \)
1 pt for answer

d. Find \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to \infty} f(x) \). (2)

\[
\lim_{x \to 0^+} f(x) = -\infty \text{ or does not exist} \]
\[
\lim_{x \to \infty} f(x) = 0 \]

1 pt for \( \lim_{x \to 0^+} f(x) \)
1 pt for \( \lim_{x \to \infty} f(x) = 0 \)
Let $f(x)$ be given by the function $f(x) = \ln \left( x + \frac{1}{x} \right)$.

a. Show that $f'(x) = \frac{x^2 - 1}{x^3 + x}$.

$$f'(x) = \left( \frac{1}{x + \frac{1}{x}} \right) \left( 1 - \frac{1}{x^2} \right) = \left( \frac{x}{x^2 + 1} \right) \left( \frac{x^2 - 1}{x^2} \right)$$

$$= \frac{x^2 - 1}{x(x^2 + 1)} = \frac{x^2 - 1}{x^3 + x}$$

1 pt for $f'(x) = \left( \frac{1}{x + \frac{1}{x}} \right) \left( 1 - \frac{1}{x^2} \right)$

1 pt for algebra

b. Find the $x$-coordinate of the point(s) on $f$ at which the tangent line to $f$ is horizontal.

$$x^2 - 1 = 0 \Rightarrow x = \pm 1 \text{ but } x \neq -1 \text{ so } x = 1$$

1 pt for answer

c. Find the equation of the tangent line to $f(x)$ at $x = 2$.

$$f'(2) = \frac{4 - 1}{8 + 2} = \frac{3}{10}$$

$$y - \ln(2.5) = \frac{3}{10}(x - 2)$$

1 pt for $f'(2)$

1 pt for answer

d. If $g(x) = e^{2f(x)}$, find $g'(2)$.

$$g(x) = e^{2f(x)} = \left[ e^{f(x)} \right]^2 = \left[ e^{\ln \left( x + \frac{1}{x} \right)} \right]^2 = \left( x + \frac{1}{x} \right)^2$$

$$g'(x) = 2 \left( x + \frac{1}{x} \right) \left( 1 - \frac{1}{x^2} \right) \text{ or } g'(x) = \frac{2(x^2 + 1)(x^2 - 1)}{x^3}$$

$$g'(2) = \frac{2(5)(3)}{8} = \frac{30}{8} = \frac{15}{4}$$

1 pt for $g(x) = \left( x + \frac{1}{x} \right)^2$

1 pt for $g'(x)$

1 pt for answer

e. Show that $f(x)$ and $g(x)$ have horizontal tangent lines at the same $x$-value(s).

$$g'(1) = \frac{2(2)(0)}{1} = 0$$

1 pt for answer
Consider the closed curve in the xy-plane given by \( x^2 - 6x + y^3 - 12y = 11 \).

a. Show that \( \frac{dy}{dx} = \frac{6 - 2x}{3y^2 - 12} \). (2)

\[
\begin{align*}
2x - 6 + 3y^2 \frac{dy}{dx} - 12 \frac{dy}{dx} &= 0 \\
\frac{dy}{dx} (3y^2 - 12) &= 6 - 2x \\
\frac{dy}{dx} &= \frac{6 - 2x}{3y^2 - 12}
\end{align*}
\]

1 pt for implicit differentiation
1 pt for verification

b. Write an equation for the line tangent to the curve at the point (6, -1). (2)

\[
\frac{dy}{dx}_{(6,-1)} = \frac{6 - 12}{3 - 12} = \frac{-6}{-9} = \frac{2}{3}
\]

Tangent line: \( y + 1 = \frac{2}{3}(x - 6) \) or \( y = \frac{2}{3}x - 5 \)

1 pt for \( \frac{dy}{dx} = \frac{2}{3} \)
1 pt for equation

c. Find the coordinates of all points on the curve where the line tangent to the curve is vertical. (3)

Tangent lines occur when \( 3y^2 - 12 = 0 \) or \( y = \pm 2 \)

\[
\begin{align*}
y = 2: x^2 - 6x + 8 - 24 &= 11 \\
x^2 - 6x - 27 &= 0 \\
(x + 3)(x - 9) &= 0 \implies x = -3, 9 \text{ so vertical tangents at } (-3,2),(9,2)
\end{align*}
\]

\[
\begin{align*}
y = -2: x^2 - 6x - 8 + 24 &= 11 \\
x^2 - 6x + 5 &= 0 \\
(x - 1)(x - 5) &= 0 \implies x = 1, 5 \text{ so vertical tangents at } (1,-2),(5,-2)
\end{align*}
\]

1 pt for \( 3y^2 = 12 \)
1 pt for \((-3,2),(9,2)\)
1 pt for \((1,-2),(5,-2)\)
2 pts maximum for only 2 of the 4 points

d. Show that it is impossible for this curve to have a horizontal tangent along the line \( y = 4 \). (2)

Horizontal tangents occur when \( 6 - 2x = 0 \) or \( x = 3 \)

At \( (3,4) \): \( 9 - 18 + 64 - 48 = 73 - 66 = 7 \)

So \( (3,4) \) is not on \( x^2 - 6x + y^3 - 12y = 11 \)

1 pt for \( 6 - 2x = 0 \)
1 pt for showing \( (3,4) \) not on curve
Consider the closed curve in the xy-plane given by \(2x^2 - xy + y^3 + x = 9\).

a. Show that \(\frac{dy}{dx} = \frac{y - 4x - 1}{3y^2 - x}\). (2)

\[
\begin{align*}
4x - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} + 1 & = 0 \\
\frac{dy}{dx}(3y^2 - x) & = y - 4x - 1 \\
\Rightarrow \frac{dy}{dx} & = \frac{y - 4x - 1}{3y^2 - x}
\end{align*}
\]

b. Find equation(s) of all tangent lines to the curve at \(y = 1\). (4)

\[
\begin{align*}
2x^2 - xy + y^3 + x & = 9 \\
x = 2: \frac{dy}{dx} & = \frac{1 - 8 - 1}{3 - 2} = -8 \\
y = 1 = -8(x - 2) \text{ or } y = \frac{8}{5}(x + 2)
\end{align*}
\]

\[
\begin{align*}
2x^2 - x + 1 + x & = 9 \\
x = -2: \frac{dy}{dx} & = \frac{1 + 8 - 1}{3 + 2} = \frac{8}{5} \\
y = 17 - 8x
\end{align*}
\]

1 pt for implicit differentiation
1 pt for answer

1 pt for \(\frac{dy}{dx}\) at \(x = 2\)
1 pt for line at \(x = 2\)
1 pt for \(\frac{dy}{dx}\) at \(x = -2\)
1 pt for line at \(x = -2\)

1 pt for equation

1 pt for \(k\)

1 pt for \(k = -1.782\)

© 2012 www.mastermathmentor.com

Illegal to post this document on the Internet
Curves $f$ and $g$ are given by the equations below as shown in the figure to the right.

Curve $f$: $x^2 - 9\ln(2y - 1) + y^2 = 5$

Curve $g$: $x^2 + e^{y^2-1} - y = 4$

a. For curve $f$, show that $\frac{dy}{dx} = \frac{2xy - x}{9 - 2y^2 + y}$

\[
2x - \frac{9}{2y-1}\left(2\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx}\left(\frac{18}{2y-1} - 2y\right) = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{\frac{18}{2y-1} - 2y}
\]

\[
\frac{dy}{dx} = \frac{2x(2y-1)}{18 - 2y(2y-1)} = \frac{4xy - 2x}{18 - 4y^2 + 2y} = \frac{2xy - x}{9 - 2y^2 + y}
\]

b. Show that horizontal tangents to curve $f$ must occur along the $y$-axis.

\[
\frac{dy}{dx} = 0 \text{ when } 2x(2y-1) = 0
\]

For curve $f$, $y$ must be a number greater than $\frac{1}{2}$

So $2x = 0 \Rightarrow x = 0$ so horizontal tangents occur along the $y$-axis

1 pt for $\frac{dy}{dx} = 0$ when $2x(2y-1) = 0$
1 pt for $y > 0$
1 pt for explanation

1 pt for implicit differentiation
1 pt for algebra

1 pt for explanation

For curve $g$, find $\frac{dy}{dx}$.

\[
2x + e^{y^2-1}(2y)\frac{dy}{dx} - \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx}\left(1 - 2y e^{y^2-1}\right) = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{1 - 2y e^{y^2-1}}
\]

1 pt for implicit differentiation
1 pt for algebra

1 pt for explanation

d. Show that the line tangent to $f$ is the same as the line normal to curve $g$ at $(2, 1)$.

Curve $f$: $\frac{dy}{dx}_{(2,1)} = \frac{2(2)(1) - 2}{9 - 2(1^2) + 1} = \frac{4 - 2}{9 - 2 + 1} = \frac{2}{8} = \frac{1}{4}$

Curve $g$: $\frac{dy}{dx}_{(2,1)} = \frac{2(2)}{1 - 2(1)e^{1-1}} = \frac{4}{1 - 2} = -4$ so normal line has slope $\frac{1}{4}$

Since the slopes are the same and they pass through the same point, the lines are the same.

1 pt for $\frac{dy}{dx}_{(2,1)}$ for $f$
1 pt for $\frac{dy}{dx}_{(2,1)}$ for $g$
1 pt for explanation
Let \( f(x) \) be given by the function

\[
    f(x) = \begin{cases} 
        9 - 4mx - (1-x)^2 & \text{if } x \leq 1 \\
        m^2x - n & \text{if } x > 1 
    \end{cases}
\]

where \( m \) and \( n \) are constants and \( m \neq 0 \).

a. Write an expression for \( n \) if \( f \) is continuous at \( x = 1 \). (1)

\[
    \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \\
    9 - 4m = m^2 - n \Rightarrow n = m^2 + 4m - 9
\]

b. Show that \( f \) cannot be continuous at \( x = 1 \) if \( n \leq -14 \). (2)

\[
    -14 = m^2 + 4m - 9 \Rightarrow m^2 + 4m + 5 = 0 \\
    m = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} \\
    \text{For any values of } n \leq -14, m \text{ will be imaginary}
\]

C. If \( f \) is differentiable at \( x = 1 \), find the values of \( m \) and \( n \). Show your reasoning. (3)

\[
    f'(x) = \begin{cases} 
        -4m - 2(1-x)(-1) & \text{if } x \leq 1 \\
        m^2 & \text{if } x > 1 
    \end{cases}
\]

\[
    \lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) \\
    -4m = m^2 \Rightarrow m^2 + 4m + 0 = 0 \Rightarrow m = -4 \\
    n = 16 - 16 - 9 = n = -9
\]

d. Using the values of \( m \) and \( n \) found in part c), determine values of \( x \) (if any) that will make \( f'(x) \) differentiable. Show your reasoning. (3)

\[
    f''(x) = \begin{cases} 
        16 + 2(1-x) & \text{if } x \leq 1 \\
        16 & \text{if } x > 1 
    \end{cases}
\]

\[
    f'''(x) = \begin{cases} 
        -2 & \text{if } x \leq 1 \\
        0 & \text{if } x > 1 
    \end{cases}
\]

\[
    \lim_{x \to 1^-} f''(x) \neq \lim_{x \to 1^+} f''(x) \text{ as } -2 \neq 0 \\
    \text{So } f'(x) \text{ is differentiable at all } x \text{-values except } x = 1
\]
Let \( f(x) \) be given by the function \( f(x) = \begin{cases} g(x) + a & \text{if } x \leq 0 \\ 3 - b \cos x & \text{if } x > 0 \end{cases} \) where \( a \) and \( b \) are constants and \( g(x) = |1 - x^2| \).

a. Determine if \( g(x) \) is differentiable at \( x = 1 \). Justify your answer. (3)

\[
g(x) = |1 - x^2| = |x^2 - 1|
\]

\[
g(x) = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ -(x^2 - 1) & \text{if } |x| < 1 \end{cases}
\]

\[
g'(x) = \begin{cases} 2x & \text{if } |x| \geq 1 \\ -2x & \text{if } |x| < 1 \end{cases}
\]

\[
\lim_{x \to 1} g(x) = 0 \quad \lim_{x \to 1^+} g(x) = 0 \quad \lim_{x \to 1^-} g'(x) = 2 \quad \lim_{x \to 1} g''(x) = -2
\]

so \( g(x) \) is continuous at \( x = 1 \) so \( g(x) \) is not differentiable at \( x = 1 \)

b. Show that \( f(x) \) is differentiable at \( x = 0 \) if \( a = 1 \) and \( b = 1 \). (3)

\[
f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ 1 - x^2 + 1 & \text{if } -1 \leq x \leq 0 \\ 3 - \cos x & \text{if } x > 0 \end{cases}
\]

\[
f'(x) = \begin{cases} -2x & \text{if } x \leq 0 \\ \sin x & \text{if } x > 0 \end{cases}
\]

\[
\lim_{x \to 0^+} f(x) = 2 \quad \lim_{x \to 0^+} f(x) = 3 - 1 = 2 \quad \lim_{x \to 0} f'(x) = 0 \quad \lim_{x \to 0} f''(x) = 0
\]

So \( f \) is continuous at \( x = 0 \) so \( f \) is differentiable at \( x = 0 \)

c. Find a relationship between \( a \) and \( b \) in order for \( f(x) \) to be continuous at \( x = 0 \). (1)

\[
f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ 1 - x^2 + 1 & \text{if } -1 \leq x \leq 0 \\ 3 - \cos x & \text{if } x > 0 \end{cases}
\]

\[
\lim_{x \to 0} f(x) = 1 + a \quad \lim_{x \to 0} f(x) = 3 - b
\]

so \( 1 + a = 3 - b \) or \( a + b = 2 \)

d. Find a relationship between \( a \) and \( b \) in order for \( f(x) \) to be differentiable at \( x = 0 \). (2)

\[
f'(x) = \begin{cases} 2x & \text{if } x < -1 \\ -2x & \text{if } -1 \leq x \leq 0 \\ b \sin x & \text{if } x > 0 \end{cases}
\]

\[
\lim_{x \to 0} f'(x) = 0 \quad \lim_{x \to 0} f''(x) = 0
\]

so \( f \) is differentiable for any value of \( b \) if \( a + b = 2 \)