Chapter 13 Exam – 50 Pts - Part 2 – 1.25 hour **Solutions**

1. Find the directional derivative of $f(x,y) = y^3 \sin 3x$ in the direction of $2\mathbf{i} - \mathbf{j}$ at the point $\left(\frac{\pi}{6}, 2, 8\right)$. (*4 pts*)

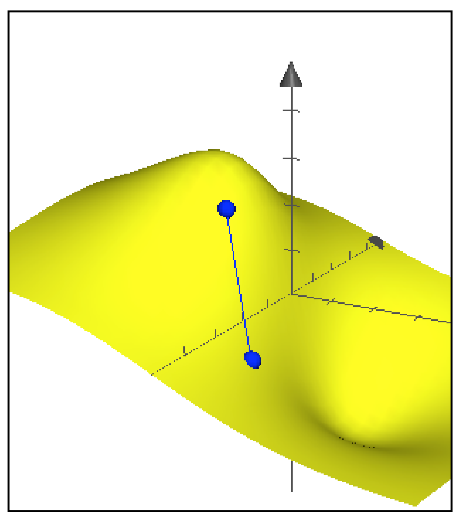
$$u = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$$
$$D_u f(x,y) = 3y^3 \cos 3x\left(\frac{2}{\sqrt{5}}\right) + 3y^2 \sin 3x\left(\frac{-1}{\sqrt{5}}\right)$$
$$D_u\left(\frac{\pi}{6}, 2\right) = 3(8)\left(\cos\frac{\pi}{2}\right)\left(\frac{2}{\sqrt{5}}\right) + 3(4)\left(\sin\frac{\pi}{2}\right)\left(\frac{-1}{\sqrt{5}}\right)$$
$$D_u\left(\frac{\pi}{6}, 2\right) = \frac{-12}{\sqrt{5}} = -5.367$$

2. Find the gradient of $f(x,y) = x^2 y + x \ln y - y + 1$ at the point (-2, 1). (*3 pts*)

$$f_x(x,y) = y + \ln y \qquad f_y(x,y) = x^2 + \frac{x}{y} - 1$$
$$\nabla f(x,y) = (y + \ln y)\mathbf{i} + \left(x^2 + \frac{x}{y} - 1\right)\mathbf{j}$$
$$\nabla f(-2,1) = \mathbf{i} + \mathbf{j}$$

3. A hiker is hiking down a mountain in the shape of $f(x,y) = \frac{12y}{x^2 + y^2 + 2}$. If he is at the point (-1, 1, 2), in what direction should he hike from that point so that he *start*s on the path of steepest descent? (*4 pts*)

$$\nabla f(x,y) = \frac{-12y(2x)}{\left(x^2 + y^2 + 2\right)^2}\mathbf{i} + \frac{12\left(x^2 + y^2 + 2\right) - 12y(2y)}{\left(x^2 + y^2 + 2\right)^2}\mathbf{j}$$
$$\nabla f(-1,1) = \frac{24}{16}\mathbf{i} + \frac{24}{16}\mathbf{j}$$
$$\nabla f(-1,1) = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \quad \text{(Path of steepest ascent)}$$
$$-\nabla f(-1,1) = -\frac{3}{2}\mathbf{i} + -\frac{3}{2}\mathbf{j} \quad \text{(Path of steepest descent)}$$



4. A different hiker is located on a mountain at the point (-2, 1, 28) whose height is given by $T(x,y) = 40 - x^2 - 8y^2$. Find the path of the hiker as he *continuously* moves in the direction of the summit (highest point on the mountain). (*6 pts*)

$\nabla T(x,y) = -4x\mathbf{i} - 8y\mathbf{j}$

The path of the particle will be $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

The tangent vector at each point is thus $r'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$

The direction of $r'(t)$ and $\nabla T(x,y)$ must be the same along the path.

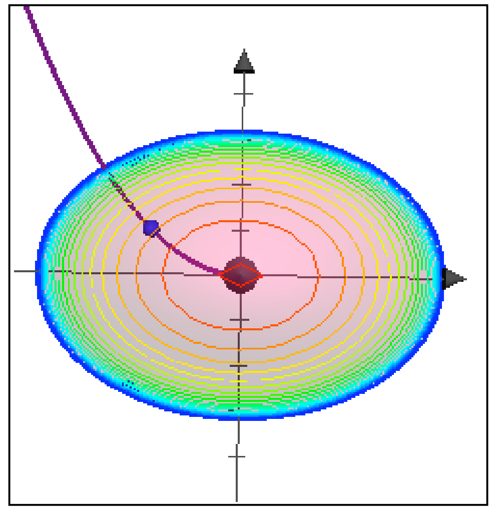
$-4x = k\frac{dx}{dt}$ and $-8y = k\frac{dy}{dt}$

$\frac{-4x}{\frac{dx}{dt}} = \frac{-8y}{\frac{dy}{dt}} \Rightarrow \frac{dy}{8y} = \frac{dx}{4x} \Rightarrow \frac{dy}{y} = \frac{2dx}{x}$

$\ln|y| = 2\ln|x| + C \Rightarrow y = Cx^2$

at point $(-2,1) \Rightarrow 1 = 4C \Rightarrow C = \frac{1}{4}$

Path is $y = \frac{1}{4}x^2$



5. Find the equation of the tangent plane to $z = 2e^{2x} + \sin y + 2$ at $x = 0, y = \pi$. (*3 pts*)

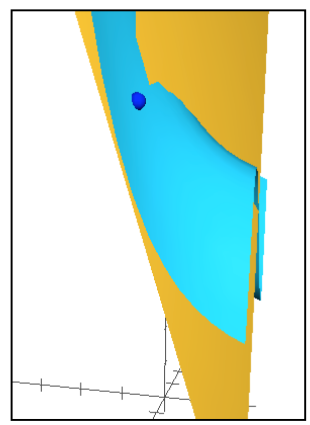
tangent plane is $f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) - (z - z_0) = 0$

$f_x = 4e^{2x}$ $\quad f_y = \cos y$ $\quad z_0 = 4$

$4(x - 0) - (y - \pi) - (z - 4) = 0$

$4x - y + \pi - z + 4 = 0$

$z = 4x - y + \pi + 4$

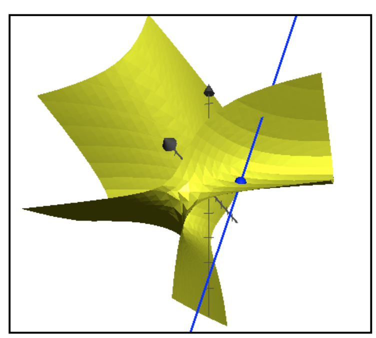


6. Find parametric equations to the normal line to $3xy - 2xz - 4yz^2 = 16$ at (3, 2, -1). (*3 pts*)

$\nabla F(x,y,z)) = (3y - 2)\mathbf{i} + (3x - 4z^2)\mathbf{j} - (2x + 8yz)\mathbf{k}$

$\nabla F(3,2,-1) = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$

$x = 3 + 4t$ $\quad y = 2 + 5t$ $\quad z = -1 + 10t$



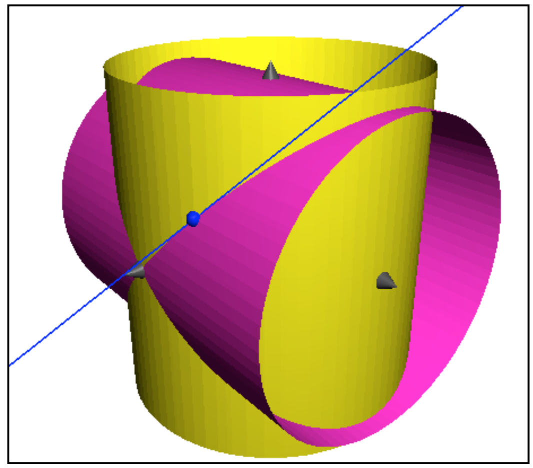
7. Find parametric equations that describe the tangent line to the intersection of the surfaces. (*5 pts*)
$x^2 + y^2 = 20$ and $x^2 + z^2 = 20$ at $(4,2,2)$.

$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j}$ $\qquad \nabla G(x,y,z) = 2x\mathbf{i} + 2z\mathbf{k}$

$\nabla F(4,2,2) = 8\mathbf{i} + 4\mathbf{j}$ $\qquad \nabla G(4,2,2) = 8\mathbf{i} + 4\mathbf{k}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 0 \\ 8 & 0 & 4 \end{vmatrix} \quad 16\mathbf{i} - 32\mathbf{j} - 32\mathbf{k} \quad (\text{tangent to both surfaces})$$

$x = 4 + 16t \qquad y = 2 - 32t \qquad z = 2 - 32t$



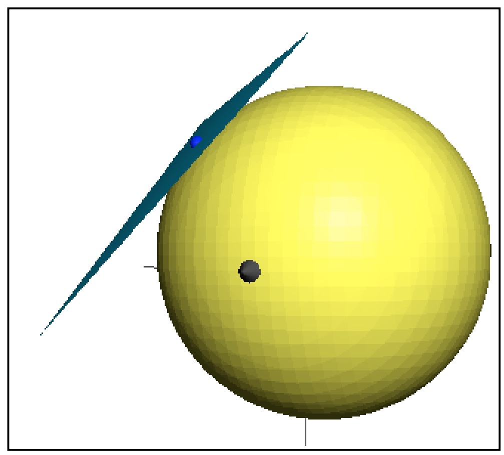
8. Find the angle of inclination of $x^2 + x + y^2 - y + z^2 - z = 30$ at $(2,-3,4)$. (*4 pts*)

$x^2 + x + y^2 - y + z^2 - z = 30$ at $(2,-3,4)$

$\nabla F(x,y,z) = (2x+1)\mathbf{i} + (2y-1)\mathbf{j} + (2z-1)\mathbf{k}$

$\nabla F(2,-3,4) = 5\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}$

$$\cos\theta = \frac{7}{\sqrt{25+49+49}} \Rightarrow \theta = 56.83^\circ$$



9. Show all work to determine points (*x, y)* of relative extrema of $z = 3x^2 - 12xy + y^3 - 20$. (*6 pts*)

$f_x = 6x - 12y = 0 \qquad f_y = -12x + 3y^2 = 0$

$x = 2y \qquad -24y + 3y^2 = 0$

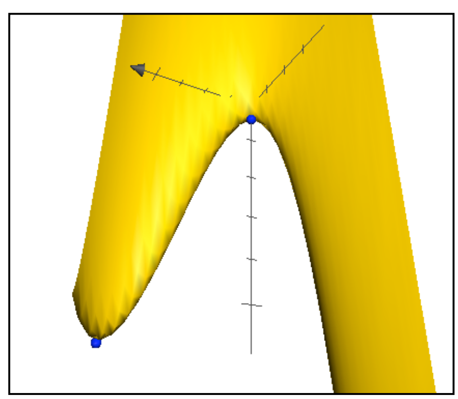
$3y(y-8) = 0 \Rightarrow y = 0, y = 8$

$f_{xx} = 6 \qquad f_{yx} = -12$

$f_{xy} = -12 \qquad f_{yy} = 6y$

at $y = 0, d < 0$ so $(0,0)$ is a saddle point

at $y = 8, d = 144 > 0$ with $f_{xx} > 0$ so $(16,8)$ is a relative minimum



10. A box with dimensions as seen in the figure to the right has the sum of its length and its girth (perimeter of a cross section) not to exceed 90 inches. Find the maximum volume of this box. (*6 pts*)

$V = xyz \qquad 2x + 2y + z = 90$

$V = xy(90 - 2x - 2y)$

$V = 90xy - 2x^2y - 2xy^2$

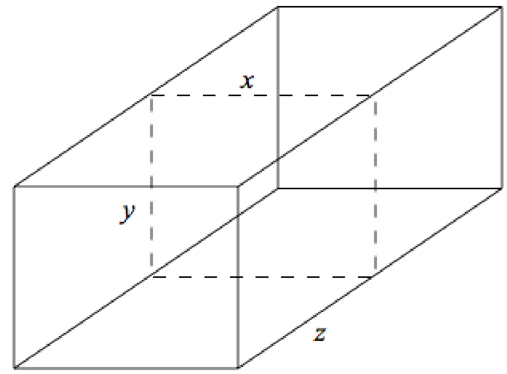
$V_x = 90y - 4xy - 2y^2 = 0 \qquad V_y = 90x - 2x^2 - 4xy = 0$

$y(90 - 4x - 2y) = 0 \qquad x(90 - 2x - 4y) = 0$

$4x + 2y = 90$ and $2x + 4y = 90$

$x = 15, y = 15, z = 30$

Maximum volume $= 6{,}750 \text{ in}^2$



11. Use LaGrange multipliers to maximize $x^2 + y^2 + z^2$ subject to $x + 2y + 4z = 42$. (*6 pts*)

$f(x,y,z) = x^2 + y^2 + z \qquad g(x,y,z) = x + 2y + 4z$

$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \qquad \nabla g = \lambda\mathbf{i} + 2\lambda\mathbf{j} + 4\lambda\mathbf{k}$

$\lambda = 2x \qquad \lambda = y \qquad \lambda = \frac{z}{2}$

$y = 2x : x + 4x + 4z = 21 \Rightarrow 5x + 4z = 42$

$y = \frac{z}{2} : x + z + 4z = 21 \Rightarrow x + 5z = 42$

$x = 2, z = 8, y = 4$

Minimum value of $x^2 + y^2 + z^2 = 84$