Section 6: Directional Derivatives

General problem: We are standing on the slope of a steep hill. We are interested in the slope of the surface. This concept by itself has no meaning. We have to decide a direction in which to walk before we can find the slope. It is possible that we could walk around the hill on a level curve so that the slope would be zero.

So we pick a point (x,y,z) on the surface in the direction of some angle θ. The slope of that surface at (x,y,z) in the direction of θ is the slope of the curve at that point. This is called a directional derivative.

Definition of Directional Derivative
If f is a function of two variables x and y and $u = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, the directional derivative is given by:
$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$

Ex 1) Find the directional derivative of $f(x,y) = 8 - x^2 - \frac{y^2}{2}$ at $(2,1)$ in the direction of $u = \cos\frac{\pi}{6}\mathbf{i} + \sin\frac{\pi}{6}\mathbf{j}$.

Ex 2) Find the dir. deriv. of $f(x,y) = x^2\cos 2y$ at $\left(1,\frac{\pi}{2},-1\right)$ in the direction of $3\mathbf{i} - \mathbf{j}$.

Ex 3) Find the directional derivative of $f(x,y) = xy + x - y$ at $(2,3)$ in the direction of $u = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$.

Ex 4) Find the directional derivative of $f(x,y) = e^y \cos x$ at $(\pi, 1, -e)$ in the direction of $u = 4\mathbf{i} - 3\mathbf{j}$.

Gradients:

A gradient is a vector-valued function of two variables. It is defined as $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$. It is read as "del f " or "grad f". Note that the gradient is a vector in the x-y plane, not space. This is similar to a directional derivative except the directional derivative is confined to a unit vector. A gradient is generalized to any vector. So an alternate form of the directional derivative is $D_u f(x,y) = \nabla f(x,y)$ as long as u is a unit vector.

Ex 5) Find the gradient of $f(x,y) = ye^x - x^2 y$ at $(2,1)$

Ex 6) Find the gradient of $f(x,y) = x^2 \text{siny}$ at $\left(2, \frac{\pi}{6}\right)$

Ex 7) Find the directional derivative of $z = 4x^2 + xy - 2y^2$ at $(1,2)$ in the direction from $(2,3)$ to $(4,1)$

Properties of the gradient (assuming that f is differentiable).

1. If $\nabla f(x,y) = 0$, then $D_u f(x,y) = 0$ for all u. (follows as $D_u f(x,y) = \nabla f(x,y)$)
2. Direction of maximum increase of f is given by $\nabla f(x,y)$. The maximum value of $D_u f(x,y) = \|\nabla f(x,y)\|$
3. Direction of minimum increase of f is given by $-a\nabla f(x,y)$. The minimum value of $D_u f(x,y) = \|\nabla f(x,y)\|$

Ex 8) Find the direction of greatest increase on $z = \dfrac{2y}{x^2 + y^2}$ from the point (2, 1).

Ex 9) A mountain climber is climbing a spire in the shape of $z = 10 - x^2 - \dfrac{y^2}{2}$ If he is at the following points, in what direction should he climb to climb at the steepest ascent?

a. $(1,2,7)$, b. $\left(-3,-1,\dfrac{1}{2}\right)$, c$\left(\dfrac{1}{2},\dfrac{-1}{2},9.625\right)$

Note that these paths will not take the climber to the top of the mountain. The path will initially be orthogonal to the contour lines of the function. In order to get to the top of the mountain, the climber needs to constantly change his path. This will be shown later.

Ex 10) A skier is skiing down a mountain in the shape of $z = \dfrac{8x}{2x^2 + y^2 + 1}$. Determine the initial path of maximum descent if he is at the points a. $(1,1,2)$, b. $\left(\dfrac{1}{2},0,2.67\right)$.