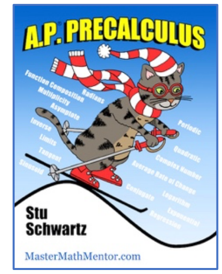
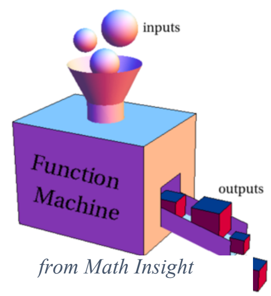
Topic 1.1 – Functions and Graph Behavior – Classwork



The heart of Precalculus is the study of functions. A *function* is a mathematical relationship that maps a set of input values to a set of output values. In the figure to the right, we see a function machine. Inputs go into the machine at the top and outputs come out the side.



Function Notation: When describing functions, we usually use the variables x and y. To write that y is a function of x, we call the function f and say that $y = f\left(x\right)$ which is stated as y equals f of x. While f is usually used to describe functions, other letters can be used such as g and h. And we are not constrained to x and y and sometimes use letters for variables that make sense in the context of the situation.

Since we input x and output y, we say that x is the independent variable and y is the dependent variable. The value of any output y depends on its input x.

For instance, if the number of points blood pressure falls is a function of the amount of a medication taken, we might write $P = f\left(m\right)$ where m is the amount of medication taken and P is the number of points fallen.

Example 1) For each situation, generate a function, describing the independent and dependent variable and the possible units in which they are measured.

a) The cost of an Iphone is a function of how much memory it has.

b) The time to fill a bathtub is a function of the rate that water comes from the faucet.

c) The temperature of a broiler is a function of the amount of time it is heated.

Domain and Range: In a function, the set of allowable input values is called the *domain* of the function. The set of allowable output values is called the *range* of the function.

In the examples above, the amount of memory available for the Iphone might be limited. The domain might be 16 gig, 32 gig, and 64 gig. And for each, there is a corresponding cost: So the range might be \$699, \$899, and \$999.

The bathtub function might have a domain of values between 0 and 2 gallons per minute. And it might have a corresponding range between 10 minutes and infinity.

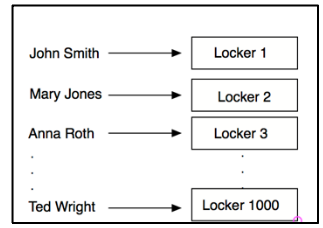
The broiler function might have a domain between 0 and infinity and a range of 0 to 550^{o} F.

Example 2) Determine the domain and range of these functions.

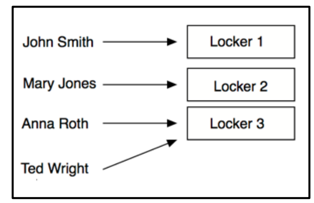
a) $f(x) = x + 1$ b) $y = x^2$ c) $g(x) = \sqrt{x-4}$ d) $y = \left|\frac{1}{x}\right|$

Function Definition: In a function, for every *x* in the domain, there can only be one and only one *y* in the range. What this says is that no *x*-value can repeat. However, it is possible for a *y*-value to repeat.

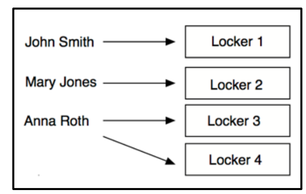
Let is see why this rule is important and why it makes sense. Suppose in a school, a locker is given to every student. We have a function that can be described in the figure to the right. The domain *x* is the set of students in the school. The range is the lockers *y*, numbered 1 – 1000. For every *x*, we have one *y*. The function is clear. If we know the student, we know his or her locker numbers.



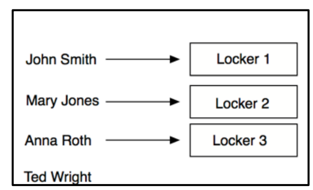
Now examine the situation to the right. In this case Anna and Ted are assigned to the same locker. So the *y*-value of 3 repeats. While this situation might not seem fair, there is no issue as far as the function is concerned. Again, we know the student, we know the locker. So repeating values of *y* are allowable. It is possible that everyone might have the same locker.



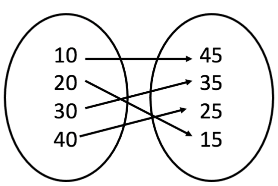
Now examine the situation to the right. In this case, Anna is assigned to 2 lockers. If we had to get something from her locker, we would not know which one contained her possessions. That's a problem. So in a function, an *x* in the domain cannot have more than one *y* in the range.

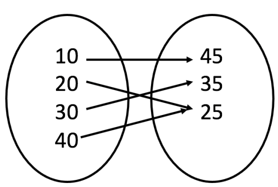


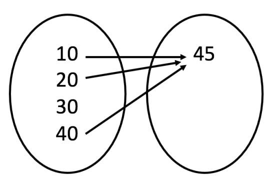
In final situation, Ted is not assigned a locker. His *x* is a valid member of the domain but there is no corresponding *y*. That obviously causes a problem. So in a function, every *x* in the domain must have a *y* in the range

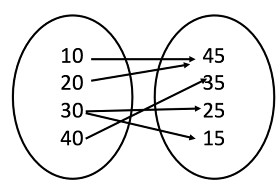


Example 3) In the following mappings, determine which are functions. If it fails to be a function, explain why.

a) 

b) 

c) 

d) 

Expressing Functions: The input and output values of a function vary in tandem (working together) according to the function rule. This rule can be expressed in 4 different ways: *verbally, analytically, numerically*, or *graphically*. Typically we start with the verbal or analytic and generate the others. Sometimes though we work in the other direction which can be harder. Each method has its strengths and weaknesses and that is why having all 4 is helpful. Throughout this course, we will shift from one method to another.

Verbally: words are used to describe the relationship. Mathematical models usually start with a verbal description of a function. Example: Older brother Steve gets \$2 more allowance then Tom.

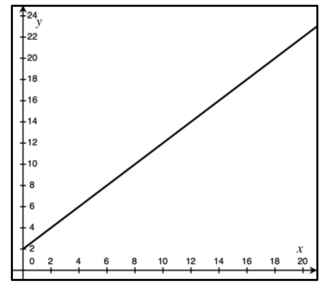
Analytically

$x =$ Tom's allowance $y =$ Steve's allowance	$y = x + 2$

Numerically: a set of possible x and y values are given

Tom	3	5	8	15	18.50
Steve	5	7	10	17	20.50

Graphically: a graph of x vs. y is shown to give a sense of the pattern of the relationship. It is important that all the important features of the relationship are shown, which we call graph behavior, covered below.



Restricting the Domain: In functions, we may want to restrict the domain of a function. Only certain values of x are permitted.

Verbally: Example, suppose a market sells deli cheese at \$6.00 a pound. But because of shortages, a customer is restricted to only 5 pounds.

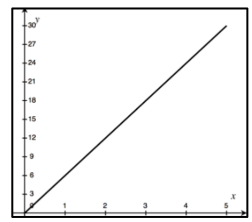
Analytically:

$x =$ Number of pounds
$y =$ Cost
$y = 6x, x \le 5$

Numerically:

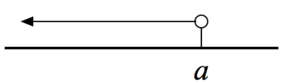
Pounds	0	0.5	1	2	5
Cost	0	3	6	12	30

Graphically:

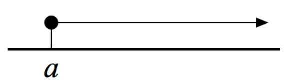


When we restrict the domain, we can describe it analytically, graphically, or with interval notation: Graphically, closed circles means that the value is included while open circles means that the value is not included. With interval notation, a bracket includes the value while a parentheses does not include the value.

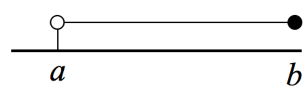
$x < a$ or $(-\infty, a)$



$x \ge a$ or $[a, \infty)$



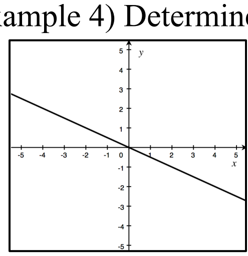
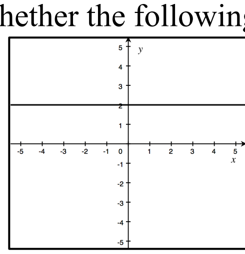
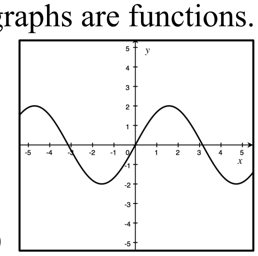
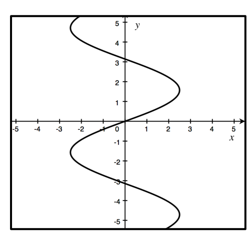
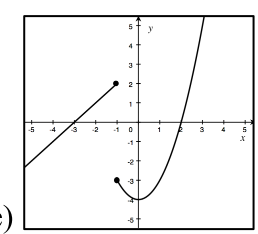
$a < x \le b$ or $(a, b]$



If both endpoints are included $[a, b]$, it is called the closed interval. If both endpoints are not included, (a, b), it is called the open interval. Since infinity or negative infinity is not a value, parentheses are always used. So $x > 4$ would be written as $(4, \infty)$ while $x \le -2$ would be written $(-\infty, -2]$.

Function Test: The test for whether a graph is a function is called the *vertical line test.* If you draw a vertical line at any x-value in its domain, the line can only intersect the graph in one point. If it intersects the graph in more than one point, it is not a function.

Example 4) Determine whether the following graphs are functions.

a)  b)  c)  d)  e) 

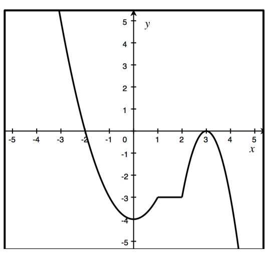
Graph Behavior: What we are most interested in is the general behavior of a graph. That includes the shape of the graph as well as several other features. We start with a definition: **increasing** and **decreasing** functions.

In general, increasing means that the function is rising as we move left to right and decreasing means the function is falling as we move left to right. However, we make a distinction in this definition.

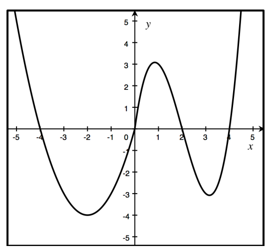
- A function is *increasing* over an interval [*a, b*] if when $b > a, f(b) \geq f(a)$. This means that if the input value increases, the output values either stay the same or increase. So it need not always increase … it just cannot decrease. If the function increases on its entire domain, we call it an increasing function.
- A function is *strictly increasing* over an interval [*a, b*] if when $b > a, f(b) > f(a)$. This means that if the input value increases, the output value increases. If the function increases on its entire domain, we call it a strictly increasing function.

For instance, suppose you are hiking up a steep hill with several flat sections. Your height above sea level is increasing over time but there are a few time intervals where it will remain the same. As long as there are no sections where you lose height, your height is an increasing function of time.

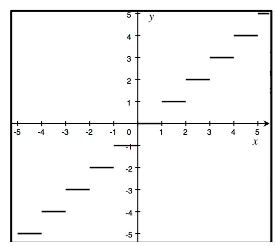
However, suppose you are purchasing deli meat at \$8 a pound. Your total cost always goes up based on the weight of the meat. This function is always increasing and can never stay the same. So cost is a strictly increasing function of time. Here are some examples:



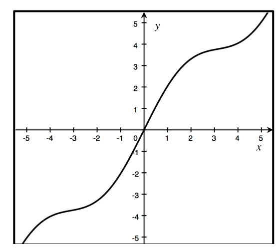
Function is increasing $0 \leq x \leq 3$



Function is strictly increasing $[-2,1],[3,\infty)$

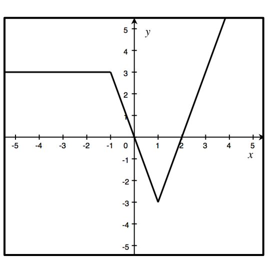


Function is increasing

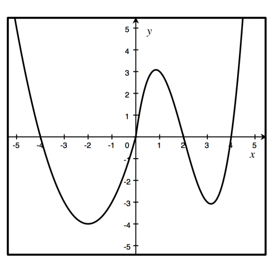


Function is strictly increasing

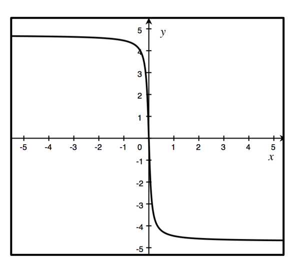
- A function is *decreasing* over an interval [*a, b*] if when $b > a, f(b) \leq f(a)$. This means that if the input value increases, the output values either stay the same or decrease. So it need not always decrease … it just cannot increase. If the function decreases on its entire domain, we call it a decreasing function.
- A function is *strictly decreasing* over an interval [*a, b*] if when $b > a, f(b) < f(a)$. This means that if the input value increases, the output value decreases. If the function decreases on its entire domain, we call it a strictly decreasing function.



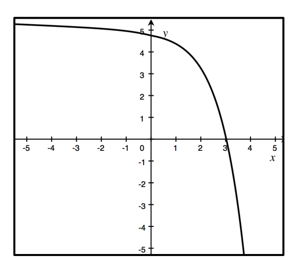
Function is decreasing $-\infty < x \leq 1$



Function is strictly decreasing $(-\infty-2],[1,3]$

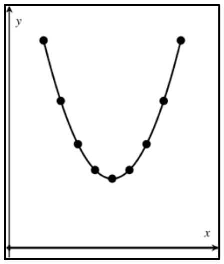


Decreasing, unclear if strictly decreasing

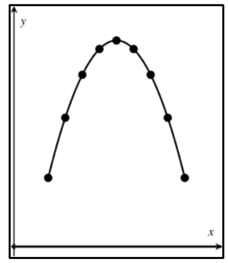


Function is strictly decreasing

When we examine functions with curves, we will define their curvature in terms of *concavity*. Concavity comes in two flavors, concave up and concave down.



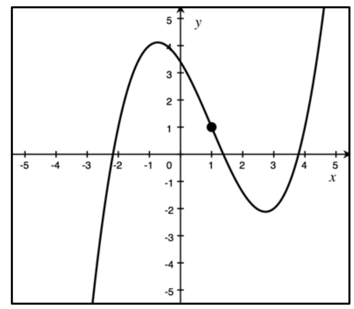
This function and any part of it is **concave up**. We say it "holds water" as water poured onto it will collect and hold.



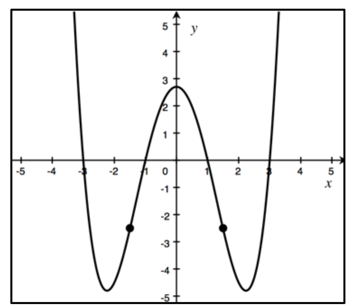
This function and any part of it is **concave down**. We say it "spills water" as water poured onto it will spill off.

Straight lines, horizontal or slanted, have no concavity.

When a curve changes from concave up to concave down, we call it an *inflection point*. The exact location of an inflection point is not obvious and it won't be until AP Calculus when you can find the specific intervals where a function is concave up, concave down, and the exact location of an inflection point. Still you should be able to recognize the approximate locations when given a graph of a function. The function to the right is concave down on $(-\infty,1)$, concave up on $(1,\infty)$ and so the inflection point appears to be at $x = 1$.

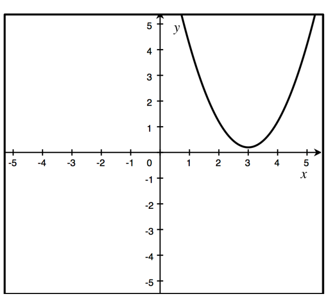


The curve to the right changes concavity several times. It is concave up on the approximate interval $(-\infty,-1.5)$ and $(1.5,\infty)$. It is concave down on $(-1.5,1.5)$ and has inflection points at $x = -1.5$ and $x = 1.5$.

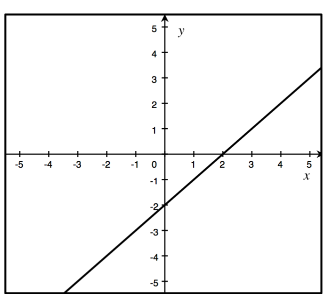


The last important aspect of a function is called its *zeros*. This is the location (the *x*-value) where the function touches or crosses the *x*-axis. We use the term "zeros" because that is where the *y*-value equals zero. Another term for zeros is *roots*.

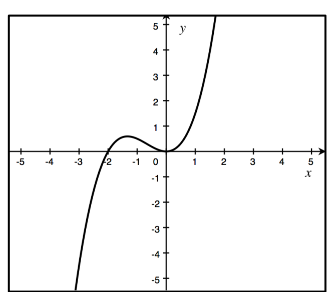
A function can have no zeros, one zero, two zeros, and an infinite number of zeros as shown below. Later in the course, we will explore analytic methods to determine the locations of the zeros but for now, like intervals of increasing, decreasing, concave up, concave down, and inflection points, we will find them by eye.



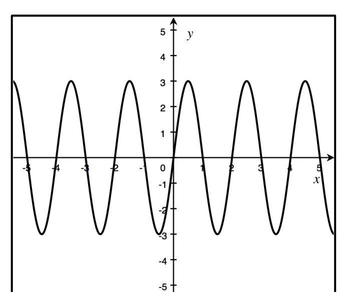
no zeros



one zero

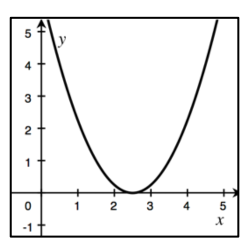


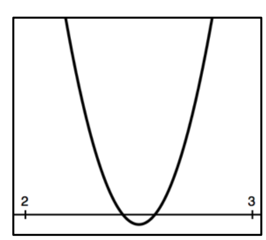
two zeros



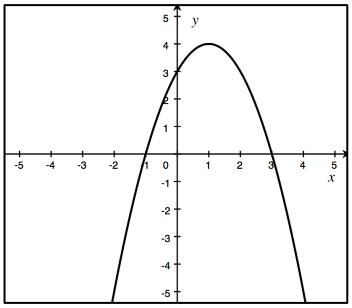
infinite zeros

Zeros are important. Sometimes a graph comes very close to the *x*-axis but it is unclear whether it touches it in 2 locations, one location, or not at all. Extreme zooming on a graphing utility must be done, or, better, the function should be explored analytically.

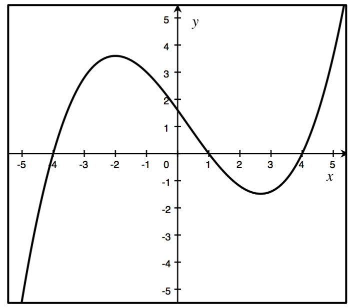




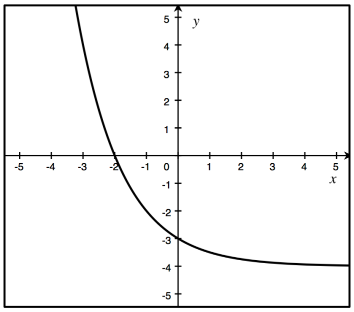
Example 5) For the following function graphs, determine intervals of increasing, strictly decreasing, concave up, concave down, and inflection points. Also find the zeros of the function.

a) 

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

b) 

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

c) 

incr:
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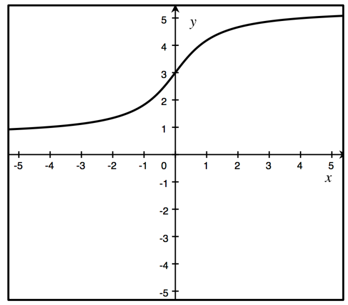
Inflection pt:
Zeros:

d) 

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

Using Graphing Utilities: This manual is set up assuming students are using TI-84's to graph functions. Any graphing utility will work. On the TI-84, a useful setting is to input the function, pressing Window to set xMin and xMax. Then pressing Zoom 0 : Fit will choose corresponding yMin and yMax to show all of the curve's important behavior.

Example 6) You are given a function and an *x*-interval to view it. Determine intervals of strictly increasing, strictly decreasing, concave up and down, inflection points (approx.) as well as the zeros of the function.

a) $y = 2^x - 4 \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

b) $y = 15 + 2x - x^2 \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

c) $y = x^3 - 4x \quad [-3,3]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

d) $y = 2x^3 - 7x^2 - 17x + 10 \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

e) $y = 5/x \quad [-10,10]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

f) $y = \left(4 - x^2\right)/\left(x^2 + 9\right) \quad [-8,8]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

Topic 1.1 – Functions and Graph Behavior – Homework

1. For each situation, generate a function, describing the independent and dependent variable as well as possible units of measure.

   a. The cost of an Uber is a function of how far the customer travels.

   b. The time to polish a floor is a function of how large it is.

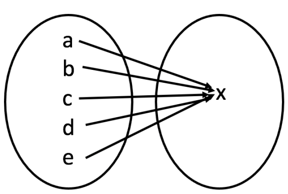
   c. The rating of a football team is a function of how many wins it has.

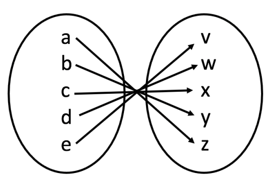
2. Determine the domain and range of these functions.

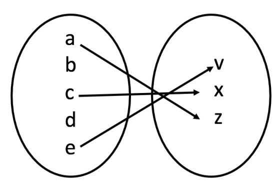
   a. $f(x) = 5x - 3$ b. $y = 5 - x^2$ c. $g(x) = \sqrt[3]{x-1}$ d. $y = \sqrt[4]{x+8}$

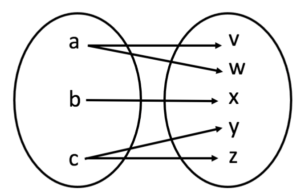
   e. $f(x) = 4^x$ f. $y = \dfrac{6}{x+4}$ g. $g(x) = 2$ h. $y = 1 - |x - 1|$

3. In the following mappings, determine which are functions. If it fails to be a function, explain why.

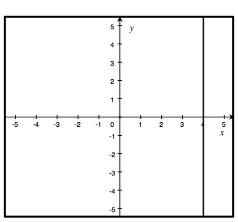
a. 

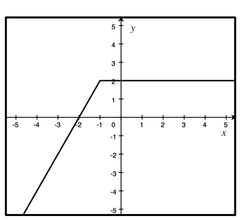
b. 

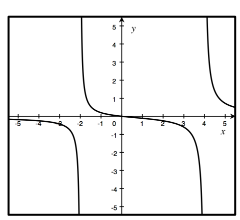
c. 

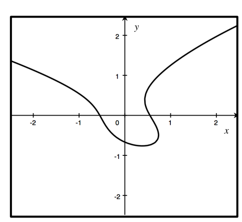
d. 

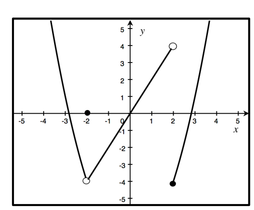
4. Determine whether the following graphs are functions.

a. 

b. 

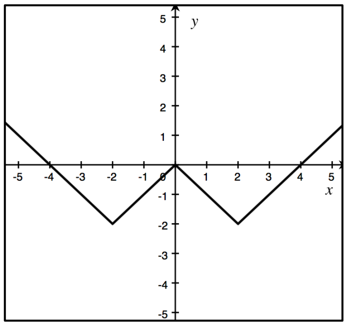
c. 

d. 

e. 

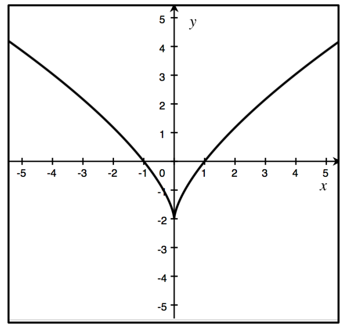
5. For the following function graphs, determine approximate intervals of increasing, decreasing, concave up, concave down, and inflection points. Also find the zeros of the function.

a.



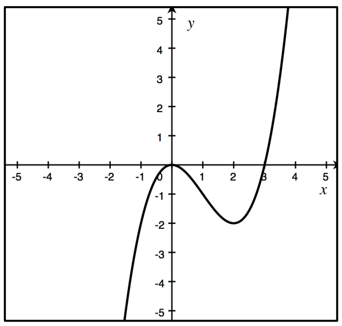
incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

b.



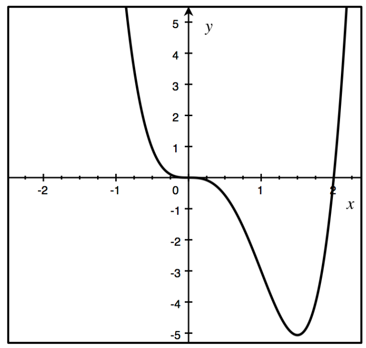
incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

c.



incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

d.



incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

6. You are given a function and an x-interval to view it. Determine intervals of increasing, decreasing, concave up and down, inflection points (approx.) as well as the zeros of the function.

a. $y = 4x \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

b. $y = 1 - x^3 \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

c. $y = 25x - x^3 \quad [-6,6]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

d. $y = \dfrac{x}{x^2 - 4} \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

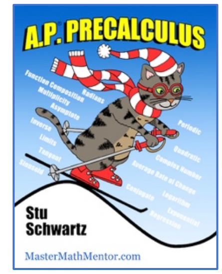
e. $y = \dfrac{x^3}{x - 1} \quad [-5,5]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

f. $y = \dfrac{x^3 + 8}{x^2 + 1} \quad [-3,3]$

incr:
decr:
concave up:
concave down:
Inflection pt:
Zeros:

Topics 1.2 – Rates of Change – Classwork



Typically as the x-values in a function change, the y-values will change as well. In precalculus, we are interested in how fast those values are changing. Sometimes, they change very slowly or occasionally, not at all. Other times, they change very quickly. We wish to quantify that change.

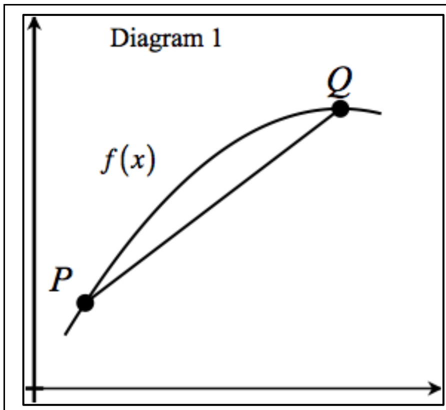
And rather than talking about that change from one point to another, we also like to examine the average change over a series of points. For instance.

- As you increase the diameter of a pizza from 10 inches to 18 inches, the average increase in price is \$1.20 per inch. That doesn't mean that every extra inch in diameter costs you an additional \$1.20. It is an average with some extra inches costing more than \$1.20 and some less.

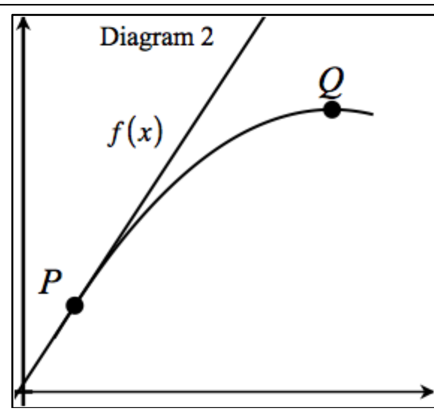
- As you increase the number from 4 people to 8, the average decrease in the time to paint an entire house is 8 hours/person. That doesn't mean that each extra person decreases the time by 8 hours. Some work faster and some slower. It is an average.

- As you increase the amount of time traveled from 0 hours to 0.5 hours, the average increase in distance is 48 miles per hour (this is called the average velocity). This doesn't mean we are always traveling at 48 mph – just an average.

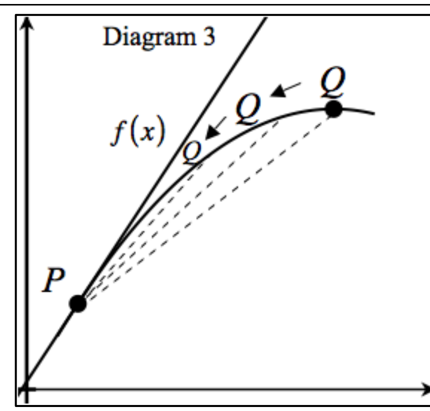
Secant Lines as Average Rate of Change



A line is drawn through points P and Q, both on $f(x)$. That line is called the **secant line** through P and Q. We call its slope the **average rate of change of** f.

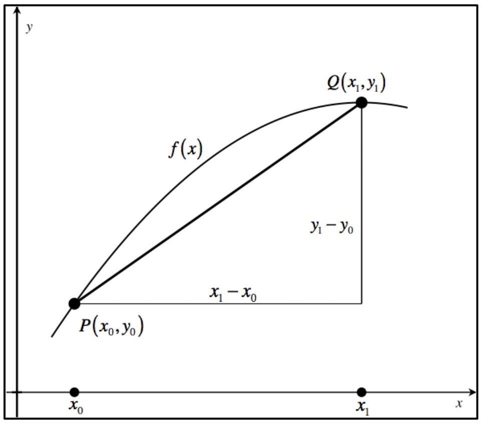


A line is drawn that touches $f(x)$ at only point P. We call its slope the **instantaneous rate of change of** f at point P.



We draw the secant line through PQ. Point Q moves along $f(x)$ towards point P. When Point Q is very close to point P, we can approximate the instantaneous rate of change at point P by using average rate of change between P and Q.

An average rate of change can be either positive or negative. A positive average rate of change indicates that as x increases, y on average increases. A plant can grow on average of 3 inches per year but that doesn't mean that at some time, the plant hasn't lost height. Similarly, a negative average rate of change indicates that as x increases, y on average decreases.



We take the two points and give them general coordinates $P(x_0, y_0)$ and $Q(x_1, y_1)$.

We draw the right triangle below the curve (the rise and the run). The length of the rise is $y_1 - y_0$ and the length of the run is $x_1 - x_0$. We now find the slope of the secant line *PQ* which we know to the average rate of change between *P* and *Q* and denote it as $m_{\text{sec}} = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_1 - y_0}{x_1 - x_0}$. Since $y = f(x)$, we can also say that

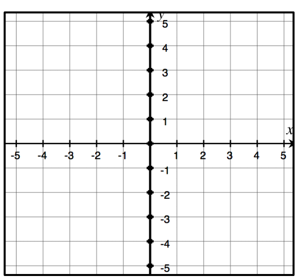
$$\boxed{m_{\text{sec}} = \text{average rate of change} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}}.$$

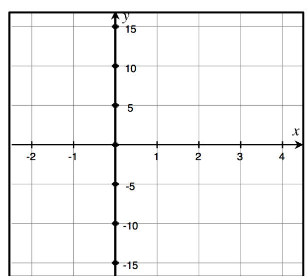
Example 1) Find the average rate of change of the given function over the interval. Draw the secant line on the curve to verify your answer.

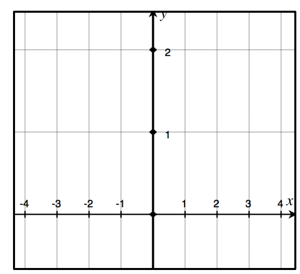
a) $f(x) = x^2 - 3 \quad [-1, 2]$

b) $g(x) = 5 - x + 3x^2 - x^3 \quad [-1, 4]$

c) $g(x) = \dfrac{4x + 8}{x^2 + 4} \quad [-4, 4]$







Instantaneous Rate of Change:

We will say that as *Q* gets closer and closer to *P*, that the average rate of change between *P* and *Q* gets closer to the instantaneous rate of change at *P*. As the interval between *P* and *Q* gets smaller, we can usually get a sense of the average rate of change at *P*.

As an example, let $f(x) = x^2$. Let us try and find the average rate of change between $x = 1$ and a value of *x* greater than but very close to 1. Complete the table. Set your calculator to maximum decimal place accuracy.

x	**2**	**1.5**	**1.1**	**1.05**	**1.01**
$f(x)$					
Rise: $f(x) - f(1)$					
Run: $x - 1$					
Average rate of change					

It appears that the instantaneous rate of change at $x = 1$ is 2. Note that this is a guess and we do not know this for sure. Only a course in calculus will determine this.

Let us use an analogy. You leave home at 10 AM on a trip and arrive at 12 noon. The trip is a total of 100 miles. How fast are you going at 11 AM? It should be obvious that we don't have a clue. We could be traveling at 60 mph. We could be stopped. We have no idea.

We cannot find the actual velocity at 11 AM (called the *instantaneous velocity*). But we can find the *average velocity* between 10 AM and any other time, using the fact that $\text{average velocity} = \dfrac{\text{total distance}}{\text{total time}}$. The average velocity between 10 AM and 12 noon is 50 mph. But again, that doesn't mean we are always traveling at 50 mph.

In the table below, let's suppose you are given the distance you traveled between 11 AM and the given time. Complete the table by calculating the time difference (in hours) and then calculate the average velocity.

Between 11 AM &	**11:30**	**11:15**	**11:10**	**11:05**	**11:01**	**11:00:30**
Distance traveled	24 miles	13 miles	10 miles	5.5 miles	0.8 miles	0.5 miles
Time duration (hrs)						
Average velocity distance/time						
Instantaneous velocity at 11 AM	???	???	???	???	???	???

While we still don't know exactly how fast we are traveleing at 11 AM, it should be clear that it should be close to 60 mph. That is because between 11 AM and 11:00:30 AM, a matter of 30 seconds, there just isn't that much time for us to radically change speeds. The closer we get to 11 AM, the closer the average velocity should be to the instantaneous velocity.

Example 2) For the following functions, find the average rate of change on the interval [*P, Q*] given. Then approximate the instantaneous rate of change at *P* using *P* + *k*. Calculators allowed.

a) $f(x) = x^2 + x \quad [1,3], k = 0.1$

Avg rate	Instantaneous rate

b) $f(x) = 4x - 1 - x^2 \quad [-3,-1], k = 0.01$

Avg rate	Instantaneous rate

c) $f(x) = x^3 - x^2 + 1 \quad [-2,3], k = 0.1$

Avg rate	Instantaneous rate

d) $f(x) = \dfrac{2}{x+1} \quad [1,4], k = 0.05$

Avg rate	Instantaneous rate

e) $f(x)=8\ \ [-5,4], k=0.1$

Avg rate	Instantaneous rate

f) $f(x)=8x\ \ [-10,12], k=0.005$

Avg rate	Instantaneous rate

In the case of the last 2 problems only, they graph lines. We show now that the average rate of change of any line over any interval is simply the slope of the line. And the instantaneous rate of change at any point using any value of *k* is also the slope of the line. Again, this is only true about lines.

g) $f(x)=mx\ \ [a,b]$

Avg rate	Instantaneous rate

Comparing Rates of Change

We can compare whether a function is changing faster at one point compared to another by approximating the instantaneous change at the two points. Since this is an approximation, there is no hard and fast rule for choosing our value of *k*. In calculus, this issue is addressed. For now, let's settle on using $k = 0.1$.

Example 3) Determine at which value of *x* the given function changes faster. Check the graph to confirm.

a) $f(x)=x^2+3x-4$, $x=0$ and $x=1$

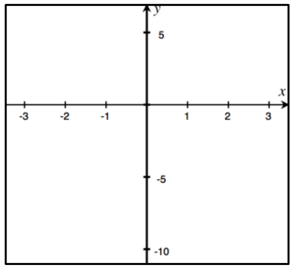
b) $f(x)=x^3-16x$, $x=-1$ and $x=2$

c) Ted puts $1,000 into a bank account that gains interest. At the end of *t* years, the amount of money in the account is given by the function $M(t)=1000\left(2^{0.05t}\right)$. Determine if his money is growing faster at the end of 1 year or 2 years. Justify your answer.

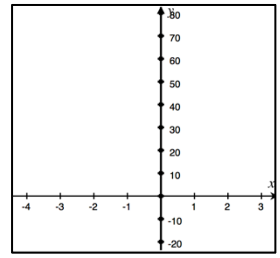
Topic 1.2 – Rates of Change – Homework

1. Find the average rate of change of the given function over the interval. Draw the secant line on the curve to verify your answer.

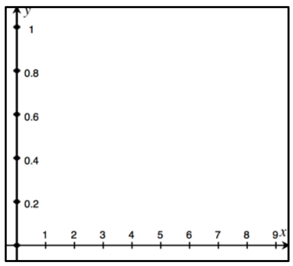
a. $f(x) = 4 - 3x^2 \quad [-2, 2]$



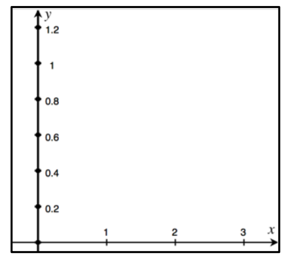
b. $g(x) = x^2 - x^3 \quad [-4, 3]$



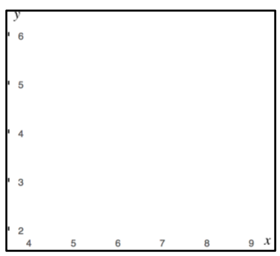
c. $g(x) = \dfrac{1}{2x-1} \quad [1, 9]$



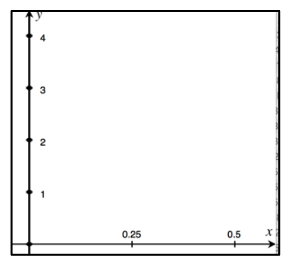
d. $h(x) = \dfrac{3x+2}{2x+3} \quad [0, 3]$



e. $f(x) = x - \sqrt{x} \quad [4, 9]$



f. $g(x) = 16^x \quad [1/4, 1/2]$



2. For the following functions, find the average rate of change on the interval [*P*, *Q*] given. Then approximate the instantaneous rate of change at *P* using *P* + *k*. Calculators allowed.

a. $f(x) = 6x - 5 \quad [2, 6], k = 0.1$

Avg rate	Instantaneous rate

b. $f(x) = x^2 + 9x - 3 \quad [-4, 4], k = 0.1$

Avg rate	Instantaneous rate

c. $f(x) = x^3 + x^2 - x - 1 \quad [0,2], k = 0.1$

Avg rate	Instantaneous rate

d. $f(x) = \dfrac{x-3}{x+3} \quad [5,9], k = 0.05$

Avg rate	Instantaneous rate

e. $f(x) = \dfrac{4}{\sqrt{x}} \quad \left[1, \frac{9}{4}\right], k = 0.01$

Avg rate	Instantaneous rate

f. $f(x) = 2^x + 2^{-x} \quad [0,2], k = 0.01$

Avg rate	Instantaneous rate

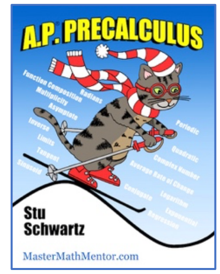
3. Using $k = 0.1$, determine at which value of x the given function changes faster. Check the graph to confirm.

a. $f(x) = 2x^2 + x - 4$, $x = 0$ and $x = 1$

b. $f(x) = -x^3 + 6x^2 + 4x - 1$, $x = -2$ and $x = 2$

c. Jer is deep sea diving and is at depth 200 feet. He must ascend very slowly or will suffer the bends. His depth D is given by $D(t) = -200 + 1.1^t$ where t is measured in minutes and negative values of D means he is below sea level. Determine if he is rising faster at a half hour or ¾ of an hour into his ascent.

Topic 1.3 – Change in Linear and Quadratic Functions – Classwork



Let's examine the rate of change of the function $f(x) = x^2 + x$ from $x = a$ to $x = b$ where a and b are any real numbers. To do that, let's start by creating a table.

x	0	1	2	3	4	5
$f(x)$	0	2	6	12	20	30

Do you see any pattern to the values of $f(x)$?

Now let's find the average rate of change (AROC) from each value of x to the next. Since the increment from x-value to x-value is 1, so we are just calculating $f(b) - f(a)$. For now, we will denote that as [*a, b*].

x	$[0,1]$	$[1,2]$	$[2,3]$	$[3,4]$	$[4,5]$
AROC	2	4	6	8	10

Do you see any pattern to the values of the AROC's.

These values are linear, meaning that they can be expressed by a line. We only need two points to find a line. Let's use (0, 2) and (1, 4). The x-value in the point is the location of the start of the first average rate of change calculation. Using $y = mx + b,\ m = \frac{4-2}{1-0} = 2$ and $y = 2x + 2$. This is the *average rate of change* formula.

Average Rate of Change Definition vs. Average Rate of Change Formula

An analogy: If we were interested in whether large objects like sofas can get around corners, we might do a series of calculations involving the measurement of the sofa as well as the width of the hallway. If we just had to do this once, that calculation is adequate. But if we were doing this repeatedly, we might want to come up with a general formula involving these parameters, and when needed, just stick them in the formula. It is more work but we then have a formula that quickly gives us the answer.

So, if you have to find the AROC between two points on a quadratic, it is best to use the AROC definition. But if we were asked to find multiple AROCs on the quadratic between 2 points, it is best to use the AROC formula.

To find the AROC between any two consecutive points [*a*, *b*] on a quadratic, you have a choice:

AROC definition = $\dfrac{f(b)-f(a)}{b-a}$	AROC formula: Generate the AROCs between $[0,1]$ and $[1,2]$ Find the equation of that line. Use that formula to find the AROC between any 2 consecutive points

Example 1) Using $f(x) = x^2 + x$, find the average rate of change between two consecutive values of x using the formula. Confirm by using the average rate of change definition.

a) 57 and 58

$\text{AROC} = 2(57) + 2 = 114 + 2 = 116$

$\text{AROC} = \dfrac{f(58) - f(57)}{58 - 57} = 3422 - 3306 = 116$

b) –100 and –99

$\text{AROC} = 2(-100) + 2 = -200 + 2 = -198$

$\text{AROC} = \dfrac{f(-99) - f(100)}{58 - 57} = 9702 - 9900 = -198$

Example 2) Is this just true with this function and with an increment of 1? Let's examine another quadratic function and a different increment. We will use $f(x) = 2x^2 - x - 4$ with an increment of 2. Let's combine the two tables and complete it. Then we will generate the linear function which represents the average rate of change of $f(x)$ with an increment of 2. Confirm by using the average rate of change definition.

a) 79 and 81

b) –1.3 and 0.7

The work we just did is not a coincidence. The method will work with any quadratic equation and with any increment. If you are asked for an AROC on an interval [a, b], $b - a$ is your increment.

Example 3) Using the general average rate formula generated above to find the specific average rate of change formula using the given $f(x)$. Then use it to find the average rate of change between the given x-values.

a) $f(x) = 4x^2 - 3x - 8 \quad [22, 23]$

b) $f(x) = 4x^2 - 3x - 8 \quad \left[-\frac{1}{2}, \frac{1}{2}\right]$

Example 4) Show that the rate of change of the average rate of change of the following quadratic functions is a constant. (Because you do not want to have to memorize the formula used in example 3, do necessary work.)

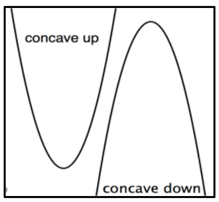
a) $f(x) = x^2 + 4x - 6$

b) $f(x) = 3x^2 - x - 10$

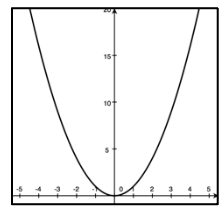
c) $f(x)=\frac{1}{2}x^2+10$

d) $f(x)=\frac{-3}{4}x^2-4x+8$

Numerical Method for Determining Concavity



In chapter 1, we described functions in terms of concavity. We used the terms concave up and concave down to describe the curvature of a graph. But at that point in time, we could only determine the concavity by looking at the graph. Let us investigate a numerical method to determine concavity.

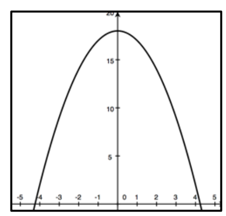


We examine the graph of $f(x)=x^2$ which we know graphs a parabola that is concave up as shown in the figure to the right.

Let us now examine the average rate of change using an increment of 1.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									
AROC									

You should notice that the average rates of change are increasing. That means the slope of the secant lines are getting larger.



We examine the graph of $f(x)=18-x^2$ which we know graphs a parabola that is concave down as shown in the figure to the right.

Let us now examine the average rate of change using an increment of 1.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									
AROC									

You should notice that the average rates of change are decreasing. That means the slope of the secant lines are getting smaller.

So we now have a method of determining the concavity of a non-linear function:

When the average rate of change over equal-length input-values is increasing for all small-length intervals, the graph of the equation is concave up.

When the average rate of change over equal-length input-values is decreasing for all small-length intervals, the graph of the equation is concave down.

Using this method of determining concavity has some issues (resolved in calculus) because there is no definite way of defining "small-length intervals". For now, let's use intervals of 0.1. Use your calculator's table feature to evaluate functions at small values of *x*.

4) Determine whether the graph of the function is concave up or concave down is the interval containing the given *x*-value. Verify graphically.

a) $f(x) = x^2 + 3x + 4, x = 0$

b. $f(x) = -10x^2 - 5,\ x = 1$

c) $f(x) = x^3 - x^2, x = 0$

d) $f(x) = \dfrac{1}{x-3}, x = 2$

5) Show that the function has a point of inflection in the interval containing the *x*-value. Verify graphically.

a) $f(x) = x - x^3, x = 0$

b) $f(x) = x^3 + x + 2, x = 0$

Topic 1.3 – Change in Linear and Quadratic Functions – Homework

1. Given the quadratic function, $f(x)$ and an increment and an interval, i) use the definition to find the average rate of change on that interval. Then ii) complete the table to find the average rate of change formula at any point x and iii) use the AROC formula to verify your answer in part i.

   a. $f(x) = x^2 + 6x, \text{increment} = 1, [34,35]$

   b. $f(x) = x^2 - 9x + 5, \text{increment} = 1, [87,88]$

   c. $f(x) = 2x^2 + 7x + 2, \text{increment} = 2, [48,50]$

   d. $f(x) = -3x^2 - 6x + 4, \text{increment} = 5, [67,72]$

   e. $f(x) = \frac{1}{4}x^2 + 8x, \text{increment} = 1, [28.5,29.5]$

2. Determine whether the graph of the function is concave up or concave down is the interval containing the given x-value. Verify graphically.

a. $f(x) = 2x^2 + 5x + 4, x = 0$

b. $f(x) = -8x^2 + 4x - 1,\ x = -1$

c. $f(x) = x^3 + 2x^2 - 3x + 1, x = 0$

d. $f(x) = \frac{-1}{x^2}, x = 2$

3. Show that the function has a point of inflection in the interval containing the x-value. Verify graphically.

a. $f(x) = x^3 + 12x, x = 0$

b. $f(x) = 2 + 6x^2 - x^3, x = 2$