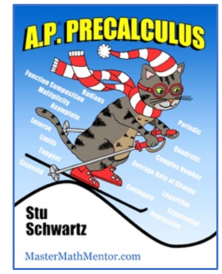
# Topic 2.1 – Arithmetic Sequences – Classwork




A *sequence* is a function whose domain is the set of positive integers, 1, 2, 3, … . Saying that a function *f* is in sequence form means that it has a first member, second member, etc., such that $a_1 = f(1), a_2 = f(2), a_3 = f(3), ..., a_n = f(n)$.

$a_1, a_2, a_3, ..., a_n ...$ are considered the terms of the sequence although on occasional it is convenient to represent the first term as $a_0$. An *infinite sequence* is a sequence that continues on forever.

The issue with infinite sequences is that continuing on forever means that there must be some logical rule to go from one term to the next. Without a rule, the sequence is useless. For instance, the sequence 2, 3, 5, 7, 11, 13, … should be familiar to you. It is the set of prime numbers. But there is no rule that has ever been found that generates the prime numbers. It is an infinite set but without a rule, the search for prime numbers continue. The largest prime number, found in November of 2022, has over 24 million digits in it.

There are two methods to express the rule for sequences: **Explicit** and **Recursive.**

- Explicit: In the explicit method, a rule as a function of $n$ is given. For instance:

  $a_n = 5n - 3 = 2, 7, 12, 17, 22, ...$ $\qquad$ $a_n = n^2 - 4n + 4 = 1, 0, -1, 4, 9, 16, .....$

  $a_n = 9 + (-1)^n = 8, 10, 8, 10, 8, 10, ...$ $\qquad$ $a_n = \dfrac{3n-2}{2n+3} = \dfrac{1}{5}, \dfrac{4}{7}, \dfrac{7}{9}, \dfrac{10}{11}, 1, ...$

- Recursive: In the recursive method, the first term $a_1$ or first couple of terms are given and the rule is given as a formula of finding the next term $a_n$ from the previous term $a_{n-1}$. For instance:

  $a_1 = 2,\ a_n = a_{n-1} + 5 = 2, 7, 12, 17, ...$ $\qquad$ $a_1 = 2,\ a_n = a_{n-1} + n = 2, 4, 7, 11, 16, 22, ...$

  $a_1 = 3,\ a_n = 4a_{n-1} - 2 = 3, 10, 38, 150, ...$ $\qquad$ $a_1 = 2,\ a_n = na_{n-1} = 2, 4, 12, 48, 240, ...$

It is possible to change from one form to another. For instance: If the explicit form is $a_n = 12n - 8$, we can write it as 4, 16, 28, 40, …. It should be apparent that the first term is 4 and succeeding terms add 12. So the recursive form is $a_1 = 4,\ a_n = a_{n-1} + 12$.

There are advantages to each method. The explicit method allows us to find any term in the sequence merely by plugging in. However, the formula might be a bit messy to use. The recursive method allows us to easily see the pattern as we go from one term to the next. However, to find any term, you must find the previous term.

So suppose you were given the sequence –1 , 0, 3 ,8, 15, 24,… and you wanted the next term.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -1 | | 0 | | 3 | | 8 | | 15 | | 24 | | **29** |
| | 1 | | 3 | | 5 | | 7 | | 9 | | **11** | |
| | | 2 | | 2 | | 2 | | 2 | | **2** | | |

Our difference method shows that the sequence was created by a quadratic formula and the next term is 29. This is in essence doing it recursively. The difference table is telling you that the next term is found by adding the next odd number to the previous term,

However, if we want the 25th term, this method would be cumbersome. Far better is the explicit formula that says $f(n) = n^2 - 2n$. The 25th term would then be $f(25) = 625 - 50 = 575$.

One of the most important sequences is the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, …. This says that the next term is the sum of the previous two terms. Expressed recursively, the Fibonacci sequence is the elegant

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}$$

$$F(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Expressed explicitly, the Fibonacci sequence is given by Binet's formula to the right: The Fibonacci sequence is important as it appears in nature in many different ways. Many sequences have real life applications.

1) Based on Census Bureau projections, the population of the United states can be modeled with the sequence $a_n = 300.76 + 2.6804n + 0.0022675n^2, n = 1,2,...,,43$ where $n$ is the number of years after 2007 and $a_n$ is measured in millions of people. a) find the expected change in population from 2024 to 2025. b) find the average rate of change in population from 2008 until 2050. c) Explain why the graph of the sequence, with points connected by a continuous curve appears linear rather than quadratic.

2) You invest the same amount of money every month into an interest-bearing savings account . The amount you will have in the account after a given time is given by the sequence $a_n = \dfrac{\text{Payment}\left[\left(1+i/12\right)^n - 1\right]}{i/12}$ where $n$ is the total number of months and $i$ is the annual interest rate. a) how much will you have in the account if your invest \$100 monthly for 5 years at 3% interest? b) compare the rate of growth of the account from year to year.

**Sigma Notation ***

Many times we wish to find the sum of a sequence. Finding the sum of the first 5 terms of 1 + 2 + 4 + …. is ambiguous. The sum differs depending on how we interpret the numbers. 1 + 2 + 4 + 8 + 16 = 31 is a logical way to interpret them with each number doubling. But 1 + 2 + 4 + 7 + 11 = 25 is also logical with the difference between numbers being 1, 2, 3, and 4.

The problem with writing such addition problems with the ellipsis (…) is that the rule for each term is not apparent. We use *Sigma Notation* for such problems using the Greek letter sigma $\sum$, which means sum. Sigma notation is used extensively in AP Calculus.

The sum of $n$ terms $a_1 + a_2 + a_3 + ... + a_n$ is written $\sum_{i=1}^{n} a_i$ where $i$ is the index of summation and $a_i$ is the $i$th term of the sum expressed explicitly. Sigma notation is a precise way to write a sum but not compute it.

1) Find the following sums.

a) $\sum_{i=1}^{8} 3$ b) $\sum_{i=1}^{6} i$ c) $\sum_{j=1}^{7} j^2$ d) $\sum_{k=-2}^{3} k^3$

Since $\sum_{i=1}^{n} a_i$ represents a summation of numbers, we can apply basic properties of addition and subtraction.

$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$ (you can factor out a *k*). $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ (write one sum as 2 sums).

e) $\sum_{i=1}^{7} 8i$ f) $\sum_{i=1}^{9}(5i-2)$ g) $\sum_{i=1}^{6}(-1)^{i+1}(2i^2-1)$ h) $\sum_{i=1}^{9}\frac{(-1)^i i}{i+1}$

**Arithmetic Sequences**

There are two types of sequences we study. In this chapter, we look at arithmetic sequences. A sequence is arithmetic if the differences between consecutive terms are the same. So the sequence $a_1, a_2, a_3, ..., a_n...$ is arithmetic is there is a number *d* such that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1} = d$. The number *d* is called the *common difference* of the arithmetic sequence.

There are three ways, the first explicit and the other 2 recursive, to find the *n*th term of an arithmetic sequence.

i. If the rule for $a_n$ is given, then you just plug the value of *n* into the rule. This is an explicit formula.
ii. $a_n = a_{n-1} + d$ where *d* is the common difference. This is a recursive formula as the previous term is needed.
iii. $a_n = a_1 + (n-1)d$, where $a_1$ is the first term, *n* is the number of terms and *d* is the common difference. This is recursive in that the first term is needed. If you know 3 variables, you can determine the 4th.

2) Find the missing variable and write the sequence.

a) $a_1 = 5, d = 3, n = 35$ b) $a_1 = 31, d = -4, n = 50$ c) $a_1 = 2, a_{50} = 639$

d) $a_1 = 45, a_{75} = -140$ e) $a_{25} = 111, d = 8$ f) $a_1 = 12, d = 3.75, a_n = 177$

It should be pointed out that the formula $a_n = a_1 + (n-1)d$ gives the formula for the *n*th term, based on knowing the first. A more general formula is $a_n = a_k + d(n-k)$ which expresses the *n* term, knowing the *k*th term. This is analogous to the general linear function with slope *m* passing through the point $(x_1, y_1)$ that can be written in the form $y = y_1 + m(x - x_1)$. The line just connects the points of an arithmetic sequence.

**Sum of Arithmetic Sequence ***

We can also find the sum of an arithmetic sequence while not actually physically adding all the terms. For instance, if we were to find $\sum_{i=1}^{50} i = 1+2+3+4+...+99+100$, it would be add smart to add them out of sequence. 1 + 100, 2 + 99, 3 + 98, …, 50 + 51. That gives us 50 (101) = 5050. Hence the formula:

$$\sum_{i=1}^{n} i = 1+2+...+n = \frac{n(n+1)}{2} \text{ or more generally, } \sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

Summation problems are expressed: i) the first and *n*th term are given, ii) sigma notation is used showing the rule, iii) the terms of the series are given showing the first term and the last term (but not how many terms).

3) Find the sums:

a) $a_1 = 5, a_{12} = 60$

b) $\sum_{n=1}^{40} 7n - 2$

c) $11+18+25+32+...+270$

d) $\sum_{n=10}^{50} 0.45n + 2.6$

e) A theatre has 9 seats on the first row, 11 seats on the 2nd row, 13 seats on the 3rd row, and so on. If the theatre has 26 rows, what is its seating capacity?

4) Dave owes \$3,420 for a loan. The payback schedule is \$200 the first week, \$195 the second week, \$190, the 3rd week, and so on until the debt is paid. How many payments must Dave make?

# Topic 2.1 – Arithmetic Sequences – Homework

1. For each of the following, write the first 5 terms of the sequence.

a. $a_n = 8 + 3n$

b. $a_n = n^2 - n$

c. $a_n = \sqrt{n}$

d. $a_n = \dfrac{n-2}{n+2}$

e. $a_n = n\left(2^{n-1}\right)$

f. $a_n = \dfrac{3(-1)^n}{n}$

2. Answer the following questions.

a. In the first few days of COVID infecting the United States, government officials conjectured that the number of cases would model the sequence $a_n = n^3 \cdot 3^{(5-n)/3}$. Find the first 6 terms of this sequence to the nearest integer and explain why this conjecture turned out to be fantastically incorrect.

b. The net profits $a_n$ in millions for CVS pharmacy for years 2015 – 2020 is given by $a_n = 13.2n + 10$, where $n$ = 1 corresponds to 2015. Show that the model $b_n = 6.6n^2 + 16.6n$ is appropriate to model the cumulative profits over those years.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a_n$ | | | | | | |
| $b_n$ | | | | | | |

c. Dave does small jobs in his building. Over a period of a week, his total pay is according to the formula $a_n = -5n^2 + 100n + 20$, where $n$ is the number of jobs he takes on. Find the first 6 terms of this sequence. Write an expression for $b_n$, representing the average pay per job and the first 6 terms of this sequence.

d. When Ted is born, his parents deposit \$10,000 in an account that earns 3.5% interest. The balance in the account after each year is given by $A_n = 10000\left(1 + 0.035/365\right)^{365n}$. Write the amount in the account on his first 5 birthdays. If his parents plan to give him the money upon high school graduation at the age of 18, how much will he have? What is the average amount of interest he received over the years?

3. For each problem, determine the sum of generating each term and calculate using the calculator:

a. $\sum_{i=1}^{6}(3i-2)$

b. $\sum_{j=1}^{60}10$

c. $\sum_{k=1}^{10}(k-1)^2$

d. $\sum_{k=1}^{10}k^2-1$

e. $\sum_{i=1}^{5}(i+1)(2i-3)$

f. $\sum_{i=1}^{7}\frac{i-2}{i+2}$

g. $\sum_{i=1}^{6}\frac{4}{i^2+2}$

h. $\sum_{i=1}^{25}(-1)^i 5$

i. $\sum_{i=1}^{5}\left[i^3-(i+1)^2\right]$

4. Find the missing variable in the arithmetic sequence and write the sequence:

a. $a_1=1, d=3, n=100$

b. $a_1=100, d=-11, n=25$

c. $a_1=-25, a_{20}=260$

d. $a_1=72, a_{66}=52.5$

e. $a_{48}=-326, d=-4$

f. $a_1=-825, d=2.5,\ a_n=420$

5. Find the sums in he arithmetic sequences:

a. $a_1=9, a_{24}=147$

b. $\sum_{n=1}^{64}3n-5$

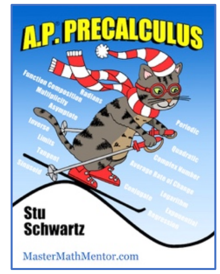
c. $-75-60-45+...+240$

d. $\sum_{n=100}^{200}10-0.2n$

e. A brick wall is in the shape of a trapezoid. The bottom row uses 37 bricks. The next row uses 36.5 bricks. Then 36, and so forth. The top row has 10 bricks. How many bricks are needed?

# Topic 2.2 – Geometric Sequences – Classwork




In section 2.1, we learned that an arithmetic sequences has a common difference between each term. In this section, we look at sequences that have a common ratio between each term. We call them *geometric sequences*.

The sequence $a_1, a_2, a_3, ..., a_n...$ is geometric if there is a number $r$ such that $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = ... = \frac{a_n}{a_{n-1}} = r, r \neq 0, r \neq 1$. The number $r$ is called the *common ratio* of the geometric sequence.
We write the geometric sequence $ar^n$.

The following sequences are geometric:

- 2, 10, 50, 250, … with $r = 5$
- the $n$th term is $3(2^n)$ as the terms are 6, 12, 24, 48, … ($r = 4$)
- the $n$th term is $(-2/3)^n$ with terms $\frac{-2}{3}, \frac{4}{9}, \frac{-8}{27}, \frac{16}{243}, ... \left(r = \frac{-2}{3}\right)$

The formula to find the $n$th term of a geometric sequence is a recursive one: $a_n = a_1 \cdot r^{n-1}$ with $a_1$ being the first term and $r$ being the common ratio. This formula has four variables and if you know any three of them, you can find the missing one.

1) Write the first four terms of the geometric series and a rule for the $n$th term.

a) $a_1 = 128, a_{k+1} = \frac{1}{2}a_k$

b) $a_1 = 729, a_{k+1} = \frac{-1}{3}a_k$

c) $a_1 = 12, a_{k+1} = \frac{3}{2}a_k$

2) Find the missing variable in the geometric sequence and write the first few terms of the sequence.

a) $a_1 = 3, r = 2, n = 8$

b) the 9th term of 2,6,18,…

c) $a_1 = -2, a_{10} = 1024$

If you know any two terms of a geometric sequence, you can find the formula for the sequence.

d) The fifth term of a geometric sequence is 175 and the tenth term is $\frac{175}{32}$. Find the 1st term and the 15th term.

The sum of a finite geometric sequence $a_1, a_1r, a_1r^2, a_1r^3$ ..., $a_1r^{n-1}$ with ratio *r*, $r \neq 1$ is given by

$S = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$. This formula can be confusing. Write out the sequence if not given and concentrate on the first term $a_1$ and the number of terms *n* which are used in the formula.

3) Find the indicated sum. Use a calculator to crunch the numbers.

a) The first 12 terms of 48,24, 12, ...

b) $\sum_{n=1}^{10} 60(0.4)^{n-1}$

c) $\sum_{n=1}^{7} 6\left(\frac{-3}{2}\right)^{n-1}$

Many times, the index begins at *n* = 0, not *n* = 1. Again, writing out the geometric sequence and determining the first term and the number of terms makes the problem easy.

4) Find the indicated sum

a) $\sum_{n=0}^{5} 3(2)^n$

b) $\sum_{n=0}^{20} 50(0.9)^n$

c) $\sum_{n=0}^{15} 5\left(\frac{-5}{4}\right)^n$

Summing the terms of an infinite geometric sequence is called a *geometric series.* In a geometric series, the sum formula $S = a_1\left(\frac{1-r^n}{1-r}\right)$ has an *n* that is infinite. If *r* <1, then the $r^n$ term becomes smaller and smaller and using limits, it becomes zero. So the sum of a geometric series is $S = \frac{a_1}{1-r}$ if *r* < 1. If *r* $\geq$ 1, then the series does not have a sum (it is infinite).

This type of problem can cause some paradoxical situations. Suppose you shoot an arrow towards a target 16 feet away. The arrow must first travel half of the distance or 8. Then it must travel half of the remaining distance or 4. Then half of the remaining distance or 2. And so on. The total distance that the arrow travels is

8 + 4 + 2 + 1 + ½ + ¼ +…. Taking half of the remaining distance means that theoretically, the arrow never reaches the target. But $\sum_{n=0}^{\infty} 8(0.5)^n = 8+4+2+... = \frac{8}{1-0.5} = 16$.

Students balk at the notion that 0.999… = 1 or $0.\overline{9} = 1$. Most say that it is *almost* one. But mathematically speaking, it is *exactly* one. Using geometric series, we get

$$0.\overline{9} = \sum_{n=0}^{\infty} 0.9(0.1)^n = 0.9 + 0.09 + 0.009 + 0.0009 + ... = 0.99999... = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$$

As you will see when you get to calculus, some strange things occur when applying the concept of infinity.

5) Find the sum of the series. Again, focus on the first term and *r*.

a) $\sum_{n=0}^{\infty} 50(0.3)^n$ b) $\sum_{n=1}^{\infty} 100(0.75)^n$ c) $\sum_{n=0}^{\infty} 20(-0.2)^n$

An annuity is set up when depositing a certain amount of money at the beginning of each month for a period of time. Suppose the Bedi's set up a college fun for their son Daman, depositing $100 at the beginning of each month at 3% interest until he turns 18.

The first deposit, gains interest for (18)(12) = 216 months. Its balance is $100\left(1+\frac{0.03}{12}\right)^{216} = 171.49$

The second deposit, gains interest for 215 months. Its balance is $100\left(1+\frac{0.03}{12}\right)^{215} = 171.06$

The last deposit, gains interest for only 1 month. Its balance is $100\left(1+\frac{0.03}{12}\right)^{1} = 100.25$

The total balance can be expressed as $\sum_{n=1}^{216} 100\left(1+\frac{0.03}{12}\right)^n$ and by our formula:

| | |
|---|---|
| 1st term is $100(1.0025)$ with 216 terms | $S = 100(1.0025)\left[\frac{1-1.0025^{216}}{1-1.0025}\right] = \$28,665$ |

6) Find the value of the annuity above with these changes.

a) Mr. Bedi gets 4% interest

b) He puts in $50 every 2 weeks with 4% interest

# Topic 2.2 – Geometric Sequences – Homework

1. Write the first 5 terms of the geometric sequence and a rule for the *n*th term.

a. $a_1 = 2, r = 5$

b. $a_1 = 1, r = \pi$

c. $a_1 = 3, r = -\sqrt{3}$

2. Find the missing variable in the geometric sequence and write the first few terms of the sequence.

a. $a_1 = 4, r = 3, n = 7$

b. the 12th term of $3, 3\sqrt{2}, 6, 6\sqrt{2}, \ldots$

c. $a_1 = \frac{9}{8}, a_6 = \frac{-1}{216}$

3. Find the first and 10th terms of the geometric sequence

a. $a_2 = 5,\ a_5 = \frac{5}{64}$

b. $a_4 = \frac{2}{3},\ a_7 = -18$

4. Find the indicated sum. Use a calculator to crunch the numbers.

a. The first 10 terms of 80,40, 20, ...

b. $\sum_{n=1}^{8} 75(0.3)^{n-1}$

c. $\sum_{n=1}^{15} 0.1\left(\frac{-9}{4}\right)^{n-1}$

d. $\sum_{n=0}^{6} 10(3)^{n}$

e. $\sum_{n=0}^{15} 64(0.85)^{n}$

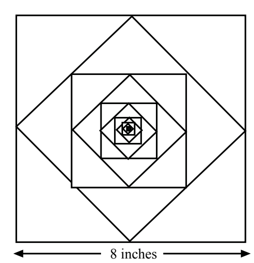
f. $\sum_{n=0}^{12} 0.01\left(\frac{-4}{3}\right)^{n}$

5. A small town with population 50,000 loses about 3% of people per year due to its proximity to a new highway. How many people are projected to live in the town in 10 years? What is the average loss per year?

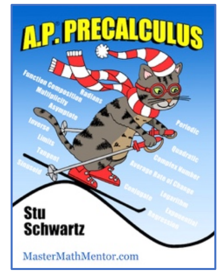
6. A lawyer joins a firm at an initial salary of $80,000 and is guaranteed a raise of 5% for every year. a. Find his projected income in his $20^{th}$ year. b. How much will he have earned with the firm after 20 years? c. Find his average salary over this time.

7. Jack is sick and takes a flu medication every hour. The doctor recommends tapering - starting with 500 mg initially and each day drop the dosage by 20%. a. If he stays on the medication for 3 weeks, how much does he take on the last day? b. How much in total will he have taken? c. If he keeps taking the medication forever, how much will he have taken?

8. The figure shows a square of side 8 in. The midpoints of the square are the vertices of an inscribed square. More squares are inscribed using the same pattern forever.
   a. Find the total area of all the squares. b. Find the total perimeter of all the squares.

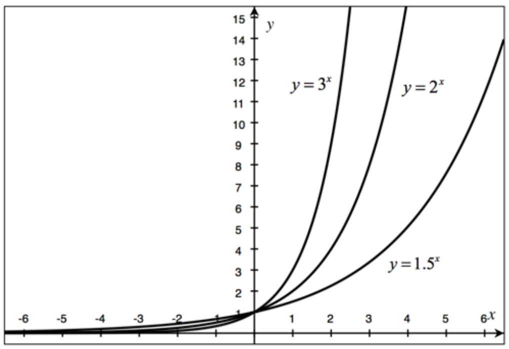


# Topic 2.3 – Exponential Functions – Classwork

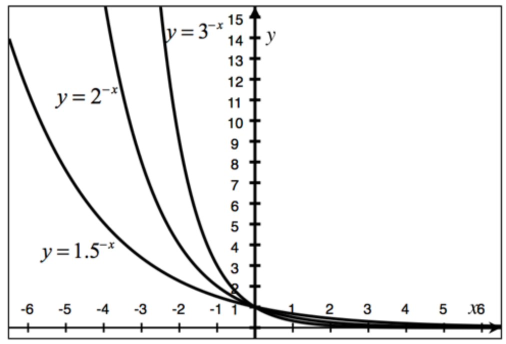



In our study of precalculus, we have examined polynomial expressions and rational expressions. What we have not examined are exponential expressions, expressions of the form $y = a \cdot b^x$. These types of expressions are very prevalent in the precalculus theatre. These sequences bear striking similarity to geometric sequences in the form of $a_n = a_1 \cdot r^{n-1}$. Sequences are a series of terms while the exponential function is a continuous curve.

An expression in the form of $y = b^x$ will graph an exponential. We call $b$ the *base* and $x$ the exponent. An *exponential graph* tends to "explode" based on the value of $x$.




$y = a^x, a > 0$




$y = a^{-x}, a > 0$

The graphs of various exponential curves are shown above. We know that the graph of $y = 1^x$ graphs a horizontal line. If $b < 0$, the graph of $y = b^x$ will not exist at certain points for instance at $x = -2$. $(-2)^{\frac{1}{2}}$ is $\sqrt{-2}$ which is imaginary. So, it only makes sense to examine functions in form of $y = b^x$, if $b > 0$, $b \neq 1$.

When $b > 1$, we get what is called a growth curve and the larger the value of $b$ is, the steeper the growth curve is. If $0 < b < 1$, we get a decay curve as shown in the 2nd graph above.

No matter what, an exponential function in the form of $y = b^x$ has certain features:

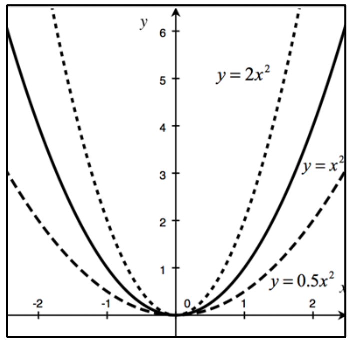
it always passes through the point $(0,1)$

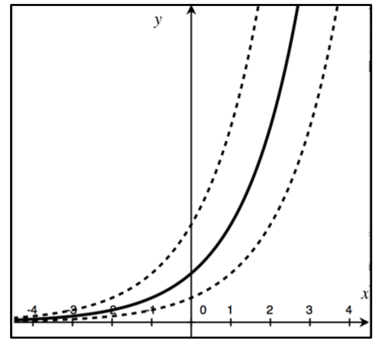
its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$

| if $b > 0$ | if $b < 0$ |
|---|---|
| its curve is increasing on $(-\infty, \infty)$ | its curve is decreasing on $(-\infty, \infty)$ |
| no relative extrema | no relative extrema |
| its curve is always concave up | its curve is always concave up |
| $\lim_{x \to \infty} f(x) = \infty$ (curve goes up to the right forever) | $\lim_{x \to -\infty} f(x) = \infty$ (curve goes up to the left forever) |
| $\lim_{x \to -\infty} f(x) = 0$ (horiz asymptote on negative $x$-axis) | $\lim_{x \to \infty} f(x) = 0$ (horiz asymptote on positive $x$-axis) |

Since exponential equations are functions, our rules for transformations of functions taught in unit 1.11 still exist, but with some interesting differences.

When we worked with parabolas, it was easy to compare it to $y = x^2$ with curves narrower or wider as can be seen in the figure to the near right. But the shape of exponentials don't have the features of a parabola as they are always getting steeper to the right or steeper to the left. In the figure to the far right, the solid curve is an exponential and the dashed curves are the same exponential shifted right and left. So, moving it left or right doesn't change its steepness as a whole. However, focusing on any one point, the steepness does change. It is clear that the the top dashed curve is increasing faster at the $y$-intercept than the other two curves.




So, when picturing transformations on exponential equations, it is best to focus on the $y$-intercept which may or may not move from (0, 1). It is easy to calculate the value of an exponential at the $y$-intercept as $x = 0$. Other than plotting points at $x = 1$ and $x = -1$, you then have to use the logic below to describe the curve.

I would suggest looking at the examples first and then switch back to this chart to see the category in which the transformation fits and the generality.

**Transformations** $y = f(x) = b^x$ (a growth curve if $b > 0$, a decay curve if $0 < b < 1$)

**Vertical Translations (Moving the curve up or down)**

$f(x) + a$ vertical shift up - translates the graph $a$ units upwards. $y$-intercept is $(0, 1 + a)$

$f(x) - a$ vertical shift down - translates the graph $a$ units downwards. $y$-intercept is $(0, 1 - a)$

**Horizontal Translations (Moving the curve right or left)**

$f(x - a)$ The curve moves right and the $y$-intercept is now $\left(0, b^{-a}\right)$ or $\left(0, \dfrac{1}{b^a}\right)$

$f(x + a)$ The curve moves left and the $y$-intercept is now $\left(0, b^{a}\right)$

**Dilations (Making the curve steeper or less steep)**

$a \cdot f(x)$ Vertical stretch - if $a > 1$, the graph is steeper at the $y$- intercept which is $(0, a)$. If $a < 1$, the graph is less steep at the $y$-intercept which is $(0, a)$.

$f(ax)$ See the example f, g and h below. $y$-intercepts do not change.

**Reflections (Obtaining a mirror image of the curve)**

$-f(x)$ Reflection - flips the graph across the $x$-axis. $y$-intercept is $(0, -a)$

$f(-x)$ Reflection across the line $y = x$. This means a growth curve changes to a decay curve and a decay curve changes to a growth curve. $y$-intercept doesn't change.

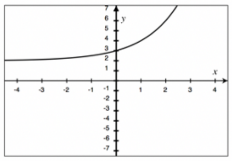
$|f(x)|$ No change as $f(x)$ is already positive.

$f|x|$ Anything to the right of the $y$-axis reflects across the $y$-axis. No asymptotes. $y$-intercept doesn't change.

Examples) If $f(x)=2^x$ , graph the following.

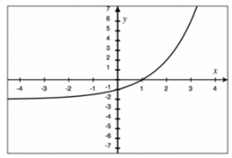
a. $f(x)+2$

$y=2^x+2$



b. $f(x)-2$

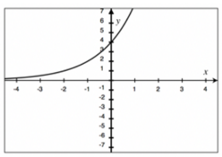
$y=2^x-2$



a) and b) are the most straightforward. They simply translate the $2^x$ curve 2 units up or 2 units down. The *y*-intercepts move to either (0, 3) or (0, –1). The steepness of the curve at the *y*-intercept remains the same as $2^x$.

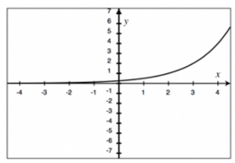
c. $f(x+2)$

$y=2^{x+2}$



d. $f(x-2)$

$y=2^{x-2}$



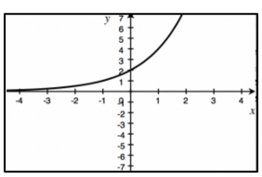
These two examples appear to be examples of horizontal translations. That would be true with polynomial or rational functions. But there is more to meet the eye here.

In c) $f(x+2)=2^{x+2}$. Our laws of exponents states that $2^{x+2}=2^x\cdot 2^2=4\left(2^x\right)$. So this is a dilation with a vertical stretch. So by translating $2^x$ two units left, the graph no longer goes through (0, 1) but (0, 4). At the *y*-intercept, the graph is increasing faster than $2^x$.

In d) $f(x-2)=2^{x-2}$. Our laws of exponents states that $2^{x-2}=2^x\cdot 2^{-2}=\frac{1}{4}\left(2^x\right)$. So this is a dilation with a vertical stretch. So by translating $2^x$ two units right, the graph no longer goes through (0, 1) but (0, ¼). At the *y*-intercept, the graph is increasing slower than $2^x$.

e. $2f(x)$

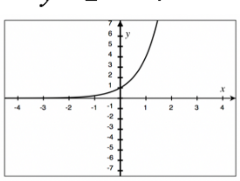
$y=2\left(2^x\right)=2^{x+1}$



In e) $2f(x)=2\left(2^x\right)$. Our laws of exponents states that $2\left(2^x\right)=2^{x+1}$. So this is a horizontal transation 1 unit to the left. The graph no longer goes through (0, 1) but (0, 2). At the *y*-intercept, the graph is increasing faster than $2^x$.

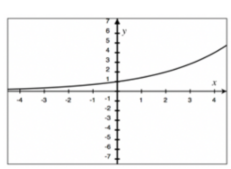
f. $f(2x)$

$y=2^{2x}=4^x$



g. $f\left(\frac{1}{2}x\right)$

$y=2^{(1/2)x}=\sqrt{2}^x$

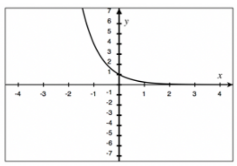


In f) $f(x)=2^{2x}$. Our laws of exponents say that $2^{2x}=\left(2^2\right)^x=4^x$. So this is a dilation making the curve steeper at the *y*-intercept which remains at (0,1).

In g) $f(x)=2^{(1/2)x}$. Our laws of exponents say that $2^{(1/2)x}=\left(2^{1/2}\right)^x=\sqrt{2}^x$. So this is a dilation making the curve less steep at the *y*- intercept which remains at (0, 1).

h. $f(-2x)$

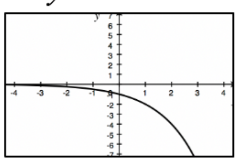
$y = 2^{-2x} = (1/4)^x$



In h) $f(x) = 2^{-2x}$. Our laws of exponents say that $2^{-2x} = \left(2^{-2}\right)^x = \left(\frac{1}{4}\right)^x$. So this changes our growth curve into a decay curve that is steeper at the $y$-intercept which remains at (0, 1). This example is why you shouldn't memorize the rules in the chart above and focus on the type of curve (growth or decay), the $y$-intercept and the curve's steepness there.

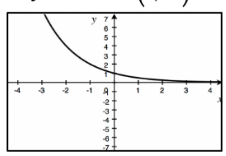
i. $-f(x)$

$y = -2^x$



j. $f(-x)$

$y = 2^{-x} = (1/2)^x$



In i), $f(x) = -2^x$. This one is easy – as it just reflects the exponential curve across the $x$-axis. The $y$-axis changes to (0, –1) and its steepness at the $y$-intercept remains the same.

In j), $f(x) = 2^{-x}$. Laws of exponents say that $2^{-x} = \left(2^{-1}\right)^x = (1/2)^x$. So this changes our growth curve into a decay curve has the same steepness at the $y$-intercept which remains at (0, 1).

When you were faced with transformations of polynomials or rational functions, you used the transformation rules and never really concerned yourself with the algebra which could be tedious. For instance, suppose you had to graph a cubic $f(x) = x^3 - 5x^2 - 2x + 24$. You factored it into $f(x) = (x-4)(x-3)(x+2)$. The zeros were at $x$ = 4, 3, and –2. Using end behavior procedures, the sketch was easy. If you now were faced with graphing $f(x-4)$, you wouldn't think of simplifying $(x-4)^3 - 5(x-4)^2 - 2(x-4) + 24$. You would reason that the function shifted 4 units to the right and the zeros had changed to $x$ = 8, 7, and 2.

However, suppose you were asked to graph $2f(x-1)$ given that $f(x) = 2^x$. You could play around with translations and dilations, but in this case, I recommend that you do the algebra. In this case $2f(x-1) = 2\left(2^{x-1}\right) = 2^{1+x-1} = 2^x$ and you get what you started with. So it is best to simply do the algebra and find the $y$-intercept, and whether it is a growth or decay curve. Plotting a point at $x$ = 1 will give you the general shape of it.

1) Given the following functions, graph the transformation.

a) $f(x) = 9^x, f\left(\frac{x+1}{2}\right)$ b) $f(x) = 16^x, f\left(\frac{2-x}{4}\right)$ c) $f(x) = 3^x, 6 - f(-x)$

**Solving Exponential Equations ***

Solving basic exponential equations can be accomplished by using the fact that if $b^x = b^y$, then $x = y$. To solve the equations below, work on one side to get the bases the same and then set the exponents equal to each other.

2) Solve for $x$.

a) $2^{x+1} = 8$ b) $3^{2x-3} = 9$ c) $5^{\frac{1}{2}x+4} = 125$ d) $3^{4x-1} = \frac{1}{3}$

e) $7^{3x+1} = \left(\frac{1}{49}\right)^{1-x}$ f) $9^{2x-4} = (27)^{x-1}$ g) $4^{5x-1} = \sqrt[3]{32}$ h) $8^{5-2x} = 1$

Later in this chapter we explore exponential equations where it is difficult to express both sides using the same base.

**Building an Exponential Model from Data ***

In algebra, you learned that it was easy to find a linear model from two points. You calculated the slope and used the point-slope equation $y - y_1 = m(x - x_1)$.

It was possible (although more work) to model a quadratic equation from any three points that do not lie on a straight line, $y = Ax^2 + Bx + C$. You plugged your 3 points in and had 3 equations in 3 unknowns, $A$, $B$, and $C$ that can be solved in a multitude of ways.

It is possible to build an exponential model from only two pieces of data. Here is the technique: The form is $y = a \cdot b^x$. You plug in both points, giving 2 equations in two unknowns. This system can be solved. Here are two examples.

Suppose your points are (3, 4) and (5, 16)

Harder: suppose your points are (4, 20) and (8, 60)

**Exponential Modeling**

When we work with data, having only 2 points leads to a decision: which one is accurate in depicting the real-life situation. The goal of modeling is to use the equation to predict the values of $y$ and any value of $x$. These two examples show that making the wrong decisions can lead to unfortunate conclusions.

3) We have two points (5, 100) and (10, 130). a) create a linear model and exponential model from the data. b) if $x = 8$, predict the value of $y$ in both models. c) If $x = 30$, predict the value of $y$ in both models. d) Use the graph to explain when the two predictions will be close to each other and the two predictions will be far away from each other.

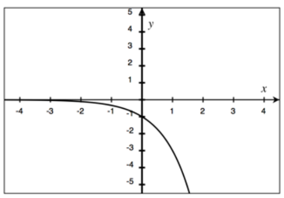
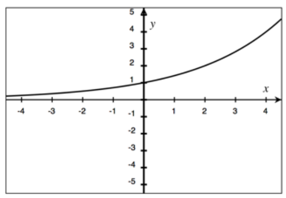
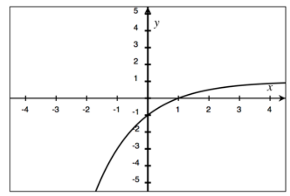
4) At 10 PM, Derek, a truck driver who has had multiple drinks finds that his blood alcohol concentration (BAC) is 0.17632 mg/mL, making him legally drunk. He stops drinking and at 12 midnight finds that his BAC has dropped to 0.07 mg/mL. While this no longer fits the criterial for legally drunk, the threshold for commercial drivers is lower, at 0.04 mg/mL.

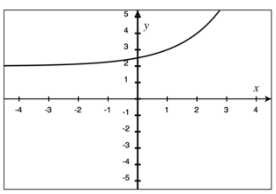
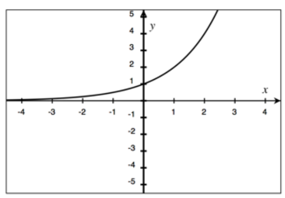
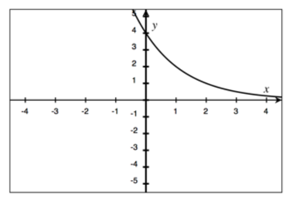
a) create a linear model and exponential model from the data and graph the curves from 10 PM until 6 AM. b) Predict the BAC in both models at 11 AM, 12.30 AM and at 1 AM. c) If he decides not to drive until his BAC is below 0.02, at what time should he leave?

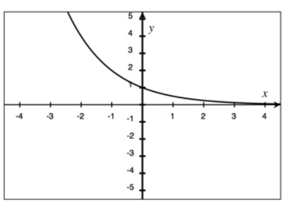
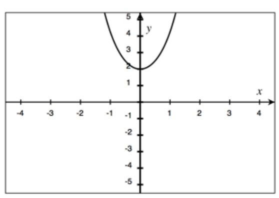
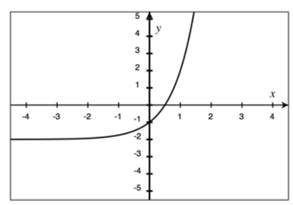
# Topic 2.3 – Exponential Functions – Homework

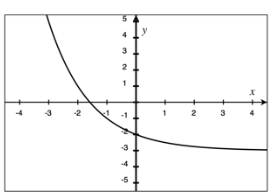
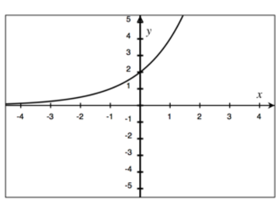
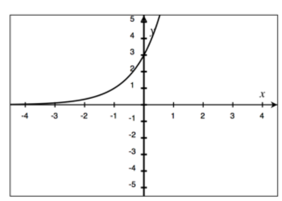
1. For each graph below, identify it by the proper equation letter. Check out certain known points (like $x = 0$) to do a trial-and-error process.

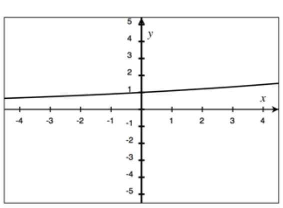
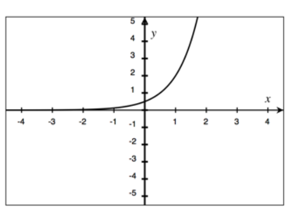
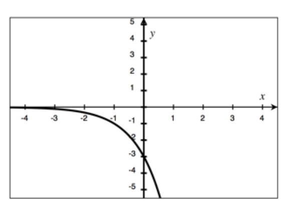
a. $y = 2^x$
b. $y = 2^{-x}$
c. $y = 2(2^x)$
d. $y = -3^x$
e. $y = -3(2^x)$
f. $y = 0.5(4^x)$
g. $y = 4^x - 2$
h. $y = 0.5^x - 3$
i. $y = -2(0.5^x) + 1$
j. $y = 3^{x+1}$
k. $y = 2^{x-1} + 2$
l. $y = 0.5^{x-2}$
m. $y = 2^{x/2}$
n. $y = 1.1^x$
o. $y = 4^x + 4^{-x}$

2. Solve for $x$

a. $2^{x-3} = 16$

b. $3^{2x-3} = 81$

c. $5^{3x-3} = 1$

d. $2^{5-2x} = \frac{1}{2}$

e. $10^{5x+6} = \frac{1}{100}$

f. $2^{4x+1} = \sqrt{2}$

g. $27^{3x+3} = 9$

h. $16^{x-3} = 8^{x-3}$

i. $25^{6-2x} = \sqrt{5}$

j. $9^{2x-4} = \left(\frac{1}{27}\right)^{x-3}$

k. $\left(\frac{1}{32}\right)^{x-7} = \left(\frac{1}{8}\right)^{x-11}$

l. $\left(\frac{1}{4}\right)^{2-2x} = \left(\sqrt[3]{2}\right)^{3x+6}$

3. Create an exponential function from the given two points and use it to predict the value of $y$ at the given $x$.

a. (0, 4) and (1,5), $x = 2$

b. (3,12) and (6,30), $x = 9$

c. (30,192) and (10,48), $x = 100$

Solve the following problems showing work.

4. Geoff just graduated law school and got a position in a firm. He needed to purchase new clothes. After 2 months, his credit card balance was \$840 and after 6 months, it was \$1,625.

a. Create a linear model and exponential model from the data and graph the curves throughout his first year. b. Predict his credit card balance after 8 months, 1 year, and 2 years. c. Are either model reasonable?

5. The Law-and-Order detectives entered a house at 11:30 PM and found a man on the floor murdered. They take his body temperature finding it to be $92.6^{o}$. The body is taken to the morgue and at 2:30 AM the body temperature is $87.2^{o}$. a. create a linear model and exponential model from the data. b. Predict what time he was killed. c. Predict the body temperature 24 hours after he was murdered. Normal body temperature is $98.6°$. d. Graph the curves through 24 hours from projected time of death.