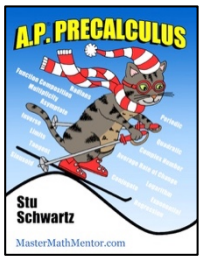
Topic 2.4 – Composition of Functions – Classwork

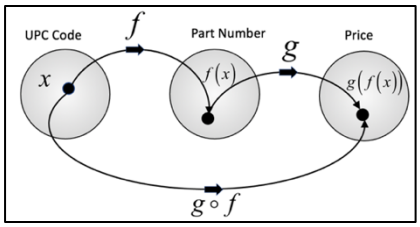


Barcodes are ubiquitous. It is difficult to find a product that does not have a barcode on it. The technical name for a barcode is a Universal Product Code or UPC. They were first used in the 70's and by the mid 80's, most larger supermarkets were using them. The UPC barcode encodes just two facts: Manufacturer and product code. The one you see here translates to "General Mills" (the manufacturer) and "21-oz Family size box of Cheerios cereal" (the product code). So in essence, the UPC code is a part number.

Once a scanner reads the codes and identifies it, that information is transferred to a computer that checks another list to find the price. That information is instantly sent back to the register and it shows on the display. The only work a supermarket teller must do is to scan the barcode. Barcodes are now even on stickers on fruit.

The advantage of barcodes is that the price can be changed instantly. Prior to the use of barcodes, a price change meant changing stickers on every box, can, or jar. Having a barcode meant that a great deal of time and money could be saved and raising the price of an item or putting it on sale happened at the computer level, not within the store. In stores now, there is just one sign above or below a product stating its price.

Mathematically speaking, barcodes are a form of **composition of functions.** You start with a function f (the barcode translating to and finding the part number). Then function g takes that part number and finds the price associated with it. We write this as $g\left(f\left(x\right)\right)$ or sometimes $\left(g\circ f\right)\left(x\right)$, or sometimes just $\left(g\circ f\right)$.

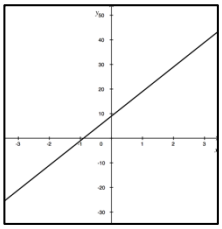


The composition of the function g with the function f is $\left(g\circ f\right)\left(x\right)=g\left(f\left(x\right)\right)$.
The composition of the function f with the function g is $\left(f\circ g\right)\left(x\right)=f\left(g\left(x\right)\right)$.
To find these, you start on the inside and work to the outside.
Composition of function are generally non *commutative*. That is $\left(g\circ f\right)\neq\left(f\circ g\right)$

Let's start off easy. If $f\left(x\right)=2x+3$ and $g\left(x\right)=5x-6$, find the following functions.

a) $\left(g\circ f\right)\left(5\right)$ b) $\left(f\circ g\right)\left(5\right)$ c) $\left(g\circ f\right)\left(x\right)$ d) $\left(f\circ g\right)\left(x\right)$

You can do these on the calculator. Define Y1 as $f\left(x\right)$ and Y2 as $g\left(x\right)$ but unselect them by placing your cursor over the equal sign and press ENTER. To look at $\left(g\circ f\right)\left(5\right)=g\left(f\left(5\right)\right)$, define Y3 as Y2(Y1). On the home screen, you can find Y3(5). You can also graph Y3 in the appropriate window and confirm that it is the same as the line $y=10x+9$.



Example 1) $f(x)=8-x^2$ and $g(x)=x+5$

a) $(g\circ f)(-2)$ b) $(f\circ g)(-2)$ c) $(g\circ f)(x)$ d) $(f\circ g)(x)$

Example 2) $f(x)=\dfrac{5}{x^2-1}$ and $g(x)=\sqrt{3x+4}$

a) $(g\circ f)(-1)$ b) $(f\circ g)(-1)$ c) $(g\circ f)(x)$ d) $(f\circ g)(x)$

Finding the domain of a function composition *: Finding the domain of a composite function involves some logic. The domain of $(g\circ f)$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

Example 3) Suppose $f(x)=\dfrac{5}{x-1}$ and $g(x)=\dfrac{3}{2x-4}$. Find the following.

a) $(f\circ g)(x)$ b) the domain of $(f\circ g)(x)$

c) $(g\circ f)(x)$ d) the domain of $(g\circ f)(x)$

Example 4) Suppose $f(x)=\sqrt{x+4}$ and $g(x)=\sqrt{5-x}$. Find the domain of $(f\circ g)(x)$ and $(g\circ f)(x)$ (confirm graphically)

Decomposing a Composite Function

Decomposing functions is identifying two functions that make up a composite function. In general, we look for an "inner" function which is done first and an "outer" function which is done second.

For instance, if $h(x)=(2x-9)^5$, we can think of the inner function $g(x)=(2x-9)$ and the outer function $f(x)=x^5$. So $h(x)=f(g(x))$.

Example 5) Decompose the following functions. There are multiple ways to do these.

a) $h(x)=\dfrac{4}{x^2-x}$ b) $h(x)=\sqrt{5x+1}$ c) $k(x)=\left|\dfrac{x^2+5}{4-x^2}\right|$

Composition of functions comes up in many real-life applications and we can model it.

Example 6) You purchase a recliner at a furniture store. You have an SUV but it will be a tough fit into the vehicle so you decide to pay to have it delivered, a cost of $75. Sales tax in your state is 7%. a) write a sales function S that includes the price of the recliner and tax, b) a fee F function, and c) a composite function T which represents what you would have to pay. How would your composite function change if you has to pay tax on the delivery charge?

Example 7) A YouTuber can make money. An advertiser will pay $0.20 per complete view of the ad. An ad is allowed at the start of the video and one at the end and one every 8 minutes. Create an ad fee function.

Topic 2.4 – Composition of Functions – Homework

For each of the two functions f and g, find the following:

1. $f(x)=x^2$ and $g(x)=2x-8$

   a) $(g\circ f)(4)$ b) $(f\circ g)(4)$ c) $(g\circ f)(x)$ d) $(f\circ g)(x)$

2. $f(x)=\dfrac{8}{x}$ and $g(x)=2x^3$

   a) $(g\circ f)(-1)$ b) $(f\circ g)(4)$ c) $(g\circ f)(x)$ d) $(f\circ g)(x)$

3. $f(x)=|x-9|$ and $g(x)=3-x^2$

   a) $(g\circ g)(-3)$ b) $(f\circ f)(5)$ c) $(g\circ g)(x)$ d) $(f\circ g)(x)$

4. Find the composite functions for $f(x)=\dfrac{2x}{x+5}$ and $g(x)=\dfrac{3x-4}{4x-3}$ and its domain.

   a. $(f\circ g)(x)$ b. the domain of $(f\circ g)(x)$

c. $(g \circ f)(x)$ d. the domain of $(g \circ f)(x)$

5. Find the composite functions for $f(x) = \frac{8}{x-4}$ and $g(x) = \sqrt{2x+4}$ and its domain.

a. $(f \circ g)(x)$ b. the domain of $(f \circ g)(x)$

c. $(g \circ f)(x)$ d. the domain of $(g \circ f)(x)$

6. Find the the composite functions for $f(x) = \frac{1}{x^3}$ and $g(x) = \frac{-1}{x+1}$, find the domain of a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ (confirm graphically)

7. Decompose the following functions. There are multiple ways to do these.

a. $h(x) = \frac{12}{2x-3}$ b. $h(x) = \frac{\sqrt{2x^2+5}}{x^2}$ c. $h(x) = (x-2)^2 - x + 2$

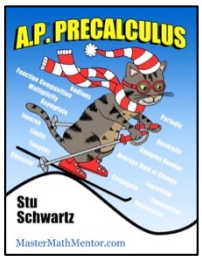
8. People wait in a 10,000 square ft. room waiting to board a cruise ship. There are 100 people waiting to board at 12 noon and that number increases by 4 people every minute. a. Write a function P for the number of people in the room at any time t. b. Write a function F for the average square footage each person has. c. Write a composite function R for the amount of square footage of room each person has as a function of time. Will the room be twice as crowded at 1:00 PM than 12:30 PM?

9. Oil is leaking from an oiler tanker in the shape of a circle. The radius after t hours is increaing at the rate of 10 feet per minute. a. Write a function R represents the rate that the area is increasing. b. Write a function A that represents the area of the circle. c) Write a composite function S that describes how the area of the spill is increasing. d. How fast is the spill increasing at 1 minute and 5 minutes?

10. Cooking a turkey is tricky. There is a fine line between a turkey being fully cooked and too dry. A celebrity chef has created a "turkey moisture formula," defining the moistness M of a cooked turkey on a scale of 0 – 10 with 10 being perfect. The formula, based on the turkey's temperature, is $M(t) = 10 - 0.02t^2 - 0.3t$ where t is the number of degrees above $165°$ F. Anything below $165°$ F is considered not fully cooked and people struggle to take the turkey out of the oven at exactly at the right time, especially because it can continue to cook when out of the oven. If a turkey stays in the oven, its temperature will be at $T(t) = 165 + 2t$ where t is the number of minutes it is left in after the turkey reached $165°$ F. If the turkey is taken out of the oven when it is $165°$ F, its temperature will be $T(t) = 165 + t/2$.

a. Write a composite function describing M when the turkey is left in oven t minutes after it reached $165°$ F.
b. Write a composite function that describes M when the turkey sits on the counter t minutes after it reached $165°$ F. Determine the moistness 5 minutes after it reached $165°$ F in both scenarios.

Topic 2.5 – Inverses – Classwork



The concept of an inverse is probably the most misunderstood concept in mathematics. Try this. To your knowledge, write the inverse of the following expressions:

5 _____ Boy_____ Dog _____ Hot dog _____

Write what you believe to be the definition of an inverse: _____

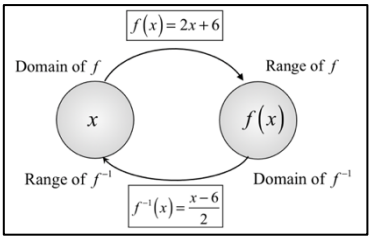
Most students think of an inverse as an opposite. And it is – but it is an opposite operation. We can take the inverse operation of going to sleep: waking up, or the inverse operation of getting dressed: getting undressed. But boys, dogs, and hot dogs are nouns and not verbs and it is meaningless to take their inverses. The concept of the opposite of a dog makes no sense. (A cat is a common answer as people tend to have dogs or cats.)

To denote the inverse of the function *f*, we denote it as f^{-1}. Do not confuse an inverse with a reciprocal. $f^{-1}(x) \neq \frac{1}{f(x)}$. This works with numbers, not functions. 5^{-1} means 1/5. Unfortunately, that notation is the same for a reciprocal. How do we tell the difference? Just as we know whether to use the word "there" or "their" by context, we know that x^{-1} refers to a reciprocal (a noun) while f^{-1} refers to a function (a verb).

The inverse function f^{-1} undoes function *f*. If we start with some number *x*, apply some process *f* to it and then immediately apply some opposite process f^{-1} to it, we get back to *x*. So, the inverse function of *f* is such that $f^{-1}\left[f(x)\right] = f\left[f^{-1}(x)\right] = x$ (within the domain of *f*).

Suppose $f(x) = 2x + 6$ having domain as integers {1, 2, 3, 4}. Then $f(x)$ consists of the points (1, 8), (2,10), (3, 12) and (4, 14). If we interchange the *x*-values and *y*-values, we get the *inverse function* $f^{-1}(x)$ consisting of the points (8, 1), (10, 2), (12, 3) and (14, 4). That can be described by the relation $f^{-1}(x) = (x-6)/2$.

Our relationship $f^{-1}\left[f(x)\right] = x$ holds (composition of functions). In the first point, function *f* takes 1 to 8. And the function f^{-1} takes 8 back to 1. This relationship is true for all points. Notice that the domain of *f* is the same as the range of f^{-1} and vice versa. That is shown analytically.



$$f^{-1}\left[f(x)\right] = \frac{2x+6-6}{2} = \frac{2x}{2} = x \qquad f\left[f^{-1}(x)\right] = 2\left(\frac{x-6}{2}\right) + 6 = x - 6 + 6 = x$$

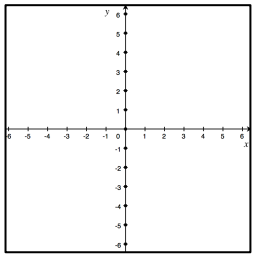
Example 1) Show that the following functions *f* and *g* are inverses:

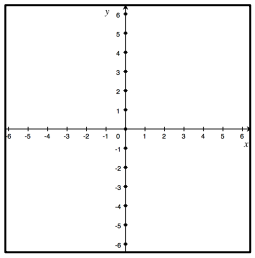
a) $f(x) = 4x - 3,\ g(x) = \frac{x+3}{4}$ b) $f(x) = \frac{x-1}{x},\ g(x) = \frac{1}{1-x}$ c) $f(x) = 8x^3 + 1,\ g(x) = \frac{\sqrt[3]{x-1}}{2}$

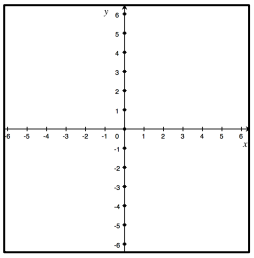
Graphical Representation of Inverses

We know that if the point (*a, b*) lies on the function, then the point (*b, a*) must lie on the inverse and vice versa. So the graph of f^{-1} is a reflection (or mirror image) of f across the line $y = x$.

Example 2) Graph the functions and inverses along with the line $y = x$ in example 1 above.

a) 

b) 

c) 

Your calculator can draw inverses. Put your function in Y1 and then use

2nd | DRAW | 8 : DrawInv | VARS | Y–VARS | 1:Function | 1:Y1

This draws the inverse (by switching X and Y-values, but does not find the inverse algebraically. Be aware that you may now have to adjust your window as the domain (Xmin and Xmax) and range (Ymin and Ymax) of the function become the range and domain of the inverse.

One-to-One Functions *

Back when functions were introduced in this course, we used an example of a function in the form of $(x, y) \Rightarrow (\text{student, locker})$. We examined several scenarios, one being that Anna was assigned both lockers one and locker two. That caused an issue as one value of *x* was assigned to two values of *y*. If we needed to collect the contents of Anna's locker, we wouldn't know which on it was in. Hence it wasn't a function.

However, we found that there was no issue if both Anna and Ted were assigned to the same locker. If we needed to collect the contents of Anna's locker, we'd know where to go. So this did not violate function rules.

However, suppose something illegal turned up in the locker that Anna and Ted shared. We wouldn't know to whom it belonged. So while sharing a locker doesn't violate function rules, it is not an ideal situation. We would prefer that everyone had his or own locker. This type of situation is called one-to-one.

One student, one locker: If we need to search a student's locker, we'd know where to go.
One locker, one student: If incriminating evidence is found in a locker, we know what student owned it.

We like to work with functions that are *one-to-one*. Meaning that not only is *f* a function but its inverse f^{-1} is also a function. In example 1a), $f(x) = 4x - 3$ and $g(x) = \dfrac{x+3}{4}$ were shown to be inverses to each other.

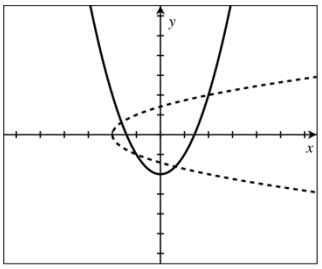
When they were graphed in 2a), both were lines and passed the vertical line test. That means that $f(x)$ is a one-to-one function.

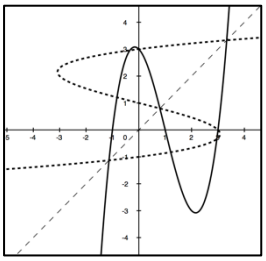
> A function is one-to-one if there is no repeating value of *y* in its range. That would mean that the domain of the inverse would not have a repeating value of *x*. If a function is not one-to-one, its inverse is not a function. If the inverse is not a function, then it is not one-to-one. Some books use the term *"f is invertible"* rather than one-to-one.

3) You are given $f(x)$ as a set of ordered pairs. Find $f^{-1}(x)$ and determine if $f(x)$ is one-to-one.

a) $f(x)=[(2,3),(3,6),(-1,-1),(0,4)]$ b) $f(x)=[(3,4),(4,2),(5,4),(0,0)]$ c) $f(x)=[(3,4),(3,5),(3,6)]$

However, functions are rarely shown using points. Suppose we are working with graphs. We see graphically that the inverse (dashed curve) of $y=x^2-2$ (solid curve) is not a function as it fails the vertical line test. So $y=x^2-2$ is not one-to-one unless we restrict its domain to $x \geq 0$.





Suppose we were given $f(x)=x^3-3x^2-x+3$. Above, we draw the function (solid) and draw the mirror image of the curve across the line $y = x$ or use the calculator to draw the inverse (dashed curve). It is clear that the inverse is not a function as it fails the vertical line test so the function $f(x)$ is not one-to-one.

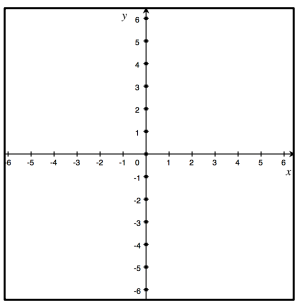
Is there a way to determine this information without going through the trouble of graphing the inverse? The answer is yes – a simple test called the *Horizontal Line Test*. If it is possible to draw a horizontal line that intersects the original function at two points, its inverse is NOT a function and thus the function is non one-to-one.

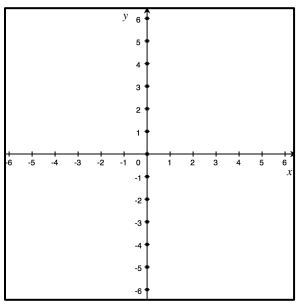
Two methods to determine whether a function *f* is one-to-one.

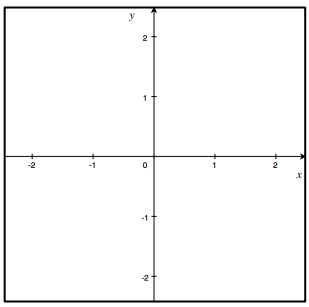
- Graph function *f*. If *f* passes the *horizontal line test*, its inverse is a function and the function is one-to one.
- Draw the inverse of f by switching x and y or using the calculator. If the *inverse* passes the *vertical line test,* the inverse is a function and thus the function *f* is one-to-one.

4) Use the horizontal line test to determine whether the given function is one-to-one.

a) $f(x)=2^x$ b) $f(x)=x^3-x$ c) $f(x)=2x^3-x^2+1$







Finding Inverse Functions Algebraically

To be clear, every function has an inverse. Whether its inverse is a function is another matter. If you restrict the domain of a function enough, it will become one-to-one. The horizontal line test gives us this information. Here are the steps to find an inverse *function* for the function *f*.

- Use the Horizontal line test to determine whether the function f is one-to-one. If it fails the test, the function is not one-to-one and thus its inverse is not a function. You can stop. Again, there is an inverse which we find below. It just isn't a function.
- Interchange the $y =$ or $f(x) =$ with $x =$. Interchange all occurrences of x with y.
- If possible, solve this new equation for y. The resulting expression is the inverse $f^{-1}(x)$.

5) For the following functions $f(x)$, find the inverse $f^{-1}(x)$. Determine if $f^{-1}(x)$ is a function.

a) $y = 2x - 1$

b) $y = 12 - 6x$

c) $y = \dfrac{x-1}{x+2}$

d) $y = x^2 - 1, x \geq 0$

e) $y = x^3 + 2$

f) $y = x^3 - x^2 - 2x - 1$

6) Given the functions $f(x) = \dfrac{x}{4} - 2$ and $g(x) = x^3$, find the following.

a) $(f^{-1} \circ g^{-1})(-8)$

b) $(g^{-1} \circ f^{-1})(14)$

c) $(f \circ g)^{-1}$

7) Your Mom tells you to go to the market and purchase a total of two pounds of two types of nuts, cashews at \$13/pound and almonds at \$9/pound. a) If x is the weight of the cashews purchased, write a function representing the cost of the entire purchase. b) Write the inverse to the cost function and describe what it means. c) Use the inverse when $x = 20$ and describe what it means.

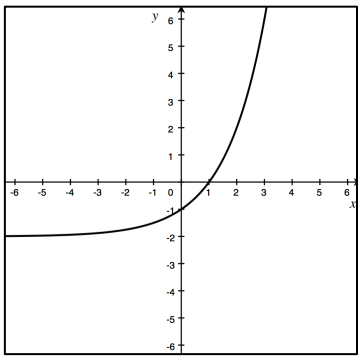
Topic 2.5 – Inverses – Homework

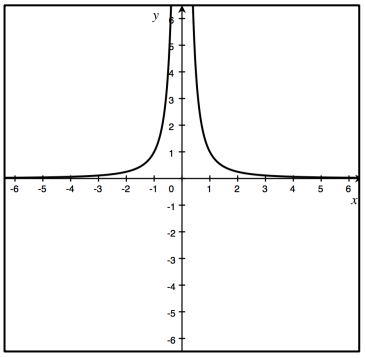
1. Show work to show that f and g are inverses.

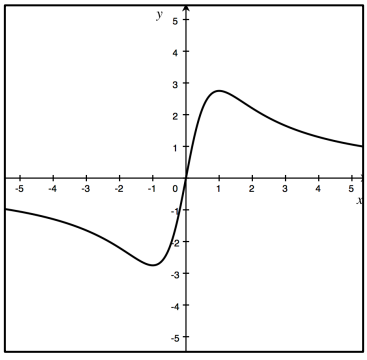
a. $f(x)=2x,\ g(x)=\frac{1}{2}x$ b. $f(x)=x^2,\ g(x)=\sqrt{x}, x\geq 0$ c. $f(x)=4x+1,\ g(x)=\frac{x-1}{4}$

d. $f(x)=\frac{x+4}{2x-5}, g(x)=\frac{5x+4}{2x-1}$

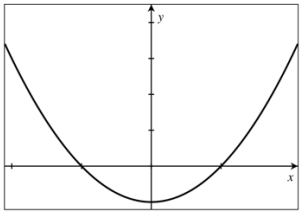
2. Sketch the inverse of each function.

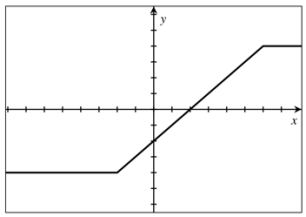
a. 

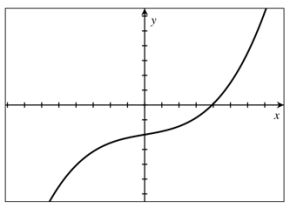
b. 

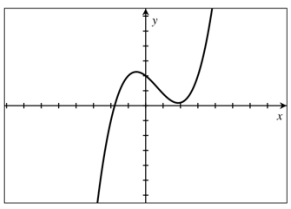
c. 

3. Which of the following functions is one-to-one? Explain.

a. 

b. 

c. 

d. 

4. For the following functions $f(x)$, find the inverse $f^{-1}(x)$. Determine if $f^{-1}(x)$ is a function.

a. $y = 6x$

b. $y = 8x + 4$

c. $y = x^2 - 5$

d. $y = x^3 + 1$

e. $y = x^2 + 4x + 4$

f. $y = \sqrt{x-1}, x \geq 1$

g. $y = \dfrac{2x-1}{x+3}$

h. $y = \sqrt{9 - x^2}, 0 \leq x \leq 3$

i. $y = \dfrac{x^2}{x^2 - 1}$

5. If $f(x) = 2x + 1$ and $g(x) = x - 3$, evaluate the following at $x = 3$.

a. $\left(f^{-1} \circ g^{-1}\right)$

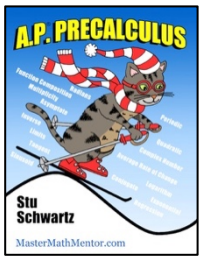
b. $\left(g^{-1} \circ f^{-1}\right)$

c. $(f \circ g)^{-1}$

d. $(g \circ f)^{-1}$

6. Jay and Marsha are planning their wedding reception. The cost of the venue is \$10,000. The cost of the meal is \$65 per beef platter and \$50 per vegetarian platter. Marsha's parents have limited the number of guests to 100 people. a) If x is the number of beef platters, write the cost as a function of x. b) Write the inverse to the cost function and describe what it means. c) Use the inverse when x = 16,200 and describe what it means.

Topic 2.6 – Logarithms – Classwork

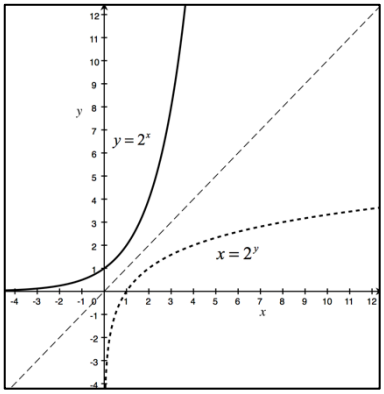


Most of your math life have been learning an operation and immediately learning the inverse operation. You started by learning addition and then immediately subtraction. You learned that $3 + 4 = 7$ and $7 - 4 = 3$ were different statements but conveyed the same information.

That was followed up with multiplication, and then division. You learned that if $2 \times 5 = 10$, then $10 \div 5 = 2$ was a different statement but conveyed the same information. However, you also found that working with zero caused issues: $2 \times 0 = 0$ but $0 \div 0 \neq 2$. So there is an exception to the blanket statement: if $P = ab$, then $b = P \div a$ and $a = P \div b$. And that is $a, b \neq 0$.

When you learned the squaring operation, you quickly also learned the inverse operation: square roots. $4^2 = 16$ and and $\sqrt{16} = 4$ conveyed the same information. Of course, this has some caveats as well because $(-4)^2 = 16$ but $\sqrt{16} \neq -4$.

So now that we have worked with exponentials as a process, we need to find an inverse to that process. For instance, when we look at the general exponential curve $y = 2^x$ (solid curve) as shown to the right, we see a one-to-one function meaning that it must have an inverse $x = 2^y$ (dashed curve) that is a mirror image of the exponential across the line $y = x$.



The issue is that at this point, we cannot solve $x = 2^y$ for y. If we want to graph it, there is no way to do so on the calculator. We need to be able to rewrite this in the form "$y =$" and for this, we need a new concept.

That concept is called a *logarith***m**. This term was first used in the mid-17th century by John Napier and was coined from the Greek: "logos" and "arithmos" meaning proportion-number.

Logarithm Definition: If $y = b^x$ then $\log_b y = x$. If $\log_b y = x$, then $y = b^x$ b is called the base and $b > 0$ and $b \neq 1$ a logarithm (abbreviated log) is simply the exponent

So, $2^5 = 32$ is equivalent to $\log_2 32 = 5$. It is simply an equivalent way of stating the same fact. Whenever you see a statement in logarithmic form, translating it to an exponential form (used to be called an anti-log) will allow you to quickly understand the relationship. For instance, if you want to find the value of $\log_8 64$, state this as $\log_8 64 = x$. Then rewrite the statement exponentially: $8^x = 64$. You should realize the answer $x = 2$ instantly. If not, write the equation with equal bases and solve.

Example 1) : Find the value of the following:

a) $\log_2 8$ b) $\log_3 81$ c) $\log_5 \frac{1}{25}$ d) $\log_4 32$

e) $\log_8 \sqrt{2}$ f) $\log_{25} \sqrt[3]{5}$ g) $\log_{36} \frac{1}{\sqrt{6}}$ h) $\log_9 1$

i) $\log_9 0$ j) $\log_3 -9$ k) $5\log_4 8$ l) $\left(\log_9 \frac{1}{3}\right)^2$

It is impossible to take logs of 0 or logs of negative numbers. $\log_3 0$ does not exist as there is no number x such that $3^x = 0$. The same is true for $\log_3 -3$. There is no number x such that $3^x = -3$. We only take logs with positive bases not equal to 1. So $y = \log_b x$ has a domain of $x > 0$.

The logarithm function can be difficult for students to grasp and utilize. Converting back and forth to exponential statements helps. Here are 5 properties of logs that you should examine and use the rationale to see that they must be true.

- $\log_b 1 = 0$ for any base $b > 0, b \neq 1$ as $b^0 = 1$
- $\log_b b = 1$ for any base $b > 0, b \neq 1$ as $b^1 = b$
- $\log_b b^x = x$ This is a property of an inverse. Start with b, raise it to the x, and then take the log base b
- $b^{\log_b x} = x$ This is a property of an inverse. Raising to a power and taking a log cancel each other]
- If $\log_b x = \log_b y$, then $x = y$ Set each equal to k. $b^k = x$ and $b^k = y \Rightarrow x = y$

Example 2) Use properties to solve these problems:

a) Solve for x: $\log_3 2x = \log_3 (4x - 6)$

b) Solve for x $\log_6 (2x + 10) = 1$

c) Simplify $\log_3 3^x$

d) Simplify $9^{\log_9 25}$

If the base is not specified, it is assumed to be 10. $\log_{10} x$ and $\log x$ have the same meaning. Log base 10 is called a *common log*. You have a LOG button on your calculator that computes a common log.

Example 3) Find the values of the following: Confirm by calculator.

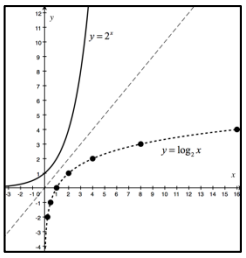
a) $\log 100$ b) $\log 1$ c) $\log \frac{1}{10}$ d) $\left(\log \frac{1}{1000}\right)^2$

e) $5\log\sqrt{10}$ f) $-3\log\sqrt[3]{10}$ g) $\log 10^4$ h) $\log 10^{\sqrt{5}}$

Graphs of Logarithmic Functions

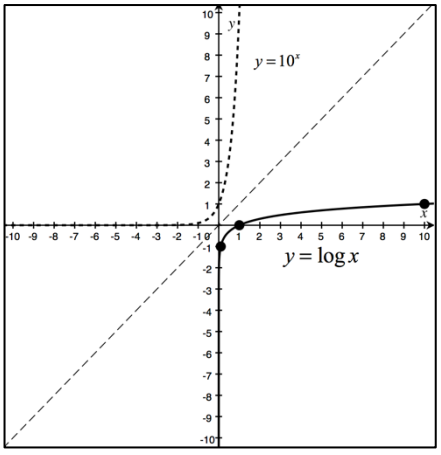
We have seen the graph of the logarithmic function as being the inverse of $y = 2^x$ which we now know that translates to $y = \log_2 x$. Confirm by plotting the appropriate points on the given graph.

x	1/4	1/2	1	2	4	8	16
$\log_2 x$	–2	–1	0	1	2	3	4



There is no dedicated key on your calculator to graph anything but common logs (base 10) for right now. In the next chapter, we will show how to graph a log with any base.

We can use the calculator to graph $y = \log x$. The calculator obviously can take logs of any positive numbers accurate to as many decimal places as we want. When asking for log 5, we get 0.69897. And that is saying that $10^{0.69897} = 5$. The graph of $y = 10^x$ and $y = \log x$ are shown together. Note that $y = \log x$ passes through the points (1,0), (10,1) and (0.1,–1). The graph of $y = \log_b x$ will have the same shape and will pass through the points (1,0), (b, 1) and (1/b, –1). Here is a comparision of $y = b^x$ and $y = \log_b x$.



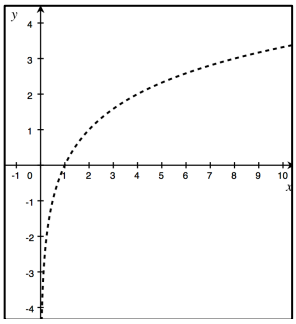
Function:	$y = b^x$	$y = \log_b x$
Domain:	$(-\infty,\infty)$	$(0,\infty)$
Range:	$(0,\infty)$	$(-\infty,\infty)$
x-intercept:	none	$(1,0)$
y-intercept:	$(0,1)$	none
Shape:	increasing, concave up	increasing, concave down
Vertical asymptote:	none	$x = 0$
Horizontal asymptote:	$y = 0$	none
Other major points:	$(1,b),(-1,1/b)$	$(b,1),(1/b,-1)$

We can do basic transformation on logarithmic curves although we usually stick to the basic vertical and horizontal shifts and reflects. The rules we establish for general functions hold for logarithmic functions.

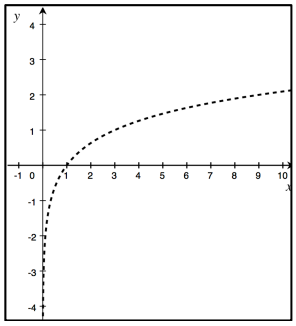
$y = a\log_b x$:	Stretches the curve vertically by a factor of a
$y = a \pm \log_b x$:	Shifts $y = \log_b x$ vertically. If $a > 0$, the shift is up. If $a < 0$, the shift is down. The vertical asymptote is not affected.
$y = \log_b(x-k)$:	Shifts $y = \log_b x$ horizontally. If $k > 0$, the shift is right. If $k < 0$, the shift is left. The vertical asymptote is shifted k units as well.
$y = -\log_b x$:	Reflects the curve across the x-axis.

Example 4) Given the dashed curve of $y = \log_b x$, sketch the transformation, plotting 3 "nice" points.

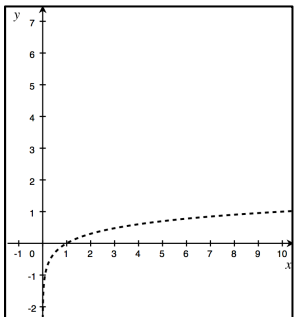
a) $y = 1 + \log_2 x$



b) $y = \log_3(x+2)$



c) $y = 10 - \log x$

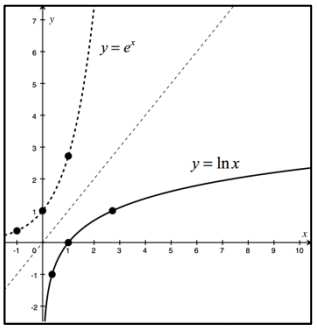


The Natural Base *e*

Even though it seems convenient to use base 10, many applications use a special irrational number as the base of an exponential. This is called *Euler's number*. Leonhard Euler (1707 - 1783) discovered this number and it is known as *e*. The value of *e* is 2.718281828…. *e* is a transcendental number which, like π and $\sqrt{2}$, continues on forever without any pattern. (Note: the 1828 in *e*, although appearing twice consecutively near the start does not appear again for a very long while. It is completely coincidental that it appears twice).

The number *e* is amazingly important and like π, shows up in very unlikely occasions. In AP Calculus, you will discover its evolution and why it is so important. The number *e* forms the basics of what are called natural logarithms of Napierian logs (after John Napier, 1550-1617, who first used them). Just as common logarithms (log) use base 10, natural logs (ln, pronounced 'el-en") use base *e*. When you wish to find the value of a log, you write the expression exponentially. You do the same thing with a natural log except that your base is now *e*.

We define the *natural logarithmic function* as $f(x) = \log_e x = \ln x, x > 0$. The ln function is the inverse of $y = e^x$ and its graph has the same properties as the ones we just studied. As we see to the right, $y = e^x$ passes through (1, *e*), (0, 1) and $(-1, 1/e)$. So $y = \ln x$ passes through (*e*, 1), (1, 0) and $(1/e, -1)$. Your calculator has an LN key as well as a LOG key. And hitting 2nd and these keys give you their inverses, e^x and 10^x.



Our properties of natural logs are the same as that of logs.

- $\ln 1 = 0$
- $\ln e = 1$
- $\ln e^x = x$ and $e^{\ln x} = x$
- if $\ln x = \ln y$, then $x = y$

Example 5) Find the value of the following:

a) $\ln e^4$ b) $\ln\frac{1}{e}$ c) $\ln\sqrt{e}$ d) $6\ln 1$

e) $9\ln e^2$ f) $\frac{1}{3}\ln\frac{1}{e^3}$ g) $\ln(-2e)$ h) $e^{\ln 5}$

i) $e\ln 0$ j) $\ln e^{\sqrt{2}}$ k) $-e\ln e$ l) $3\ln\frac{1}{\sqrt[3]{e^2}}$

6) **Applications:** We have an entire chapter devoted to exponential applications. Here are some for logarithms.

a) If I put a sum of money in the bank, the amount of time it takes to double is given by $t=\frac{\ln 2}{r}$ where r is the annual rate of interest. i) Graph this expression for possible rates up to 8%. ii) If I wanted the money to double in less than 10 years, what rate would I need? iii) What is the average rate of change of the doubling time if the interest rate goes from 3% to 3.5%

b) After taking a cough suppressant, the time in hours it takes for the medication to reduce to a given percentage in the bloodstream is given by $t=\frac{\ln(\text{Percent})}{\ln 0.85}$ or $\frac{\log(\text{Percent})}{\log 0.85}$. i) Show graphically that these two equations are equivalent. ii) Compare the amount of time it takes the medication to reduce from 80% to 70% as opposed to 20% to 10%. iii) Explain the difference in time.

Topic 2.6 – Logarithms – Homework

1. Find the value of the following expressions.

a. $\log_4 256$

b. $\log_2 8$

c. $\log_8 2$

d. $\log_5 125$

e. $\log_9 27$

f. $\log_7 1$

g. $\log_{25} 5$

h. $\log_8 16$

i. $\log_{\sqrt{3}} 27$

j. $\log_{\frac{1}{5}} \frac{1}{125}$

k. $-5\log_{12} 12$

l. $10^{\log 29}$

m. $\log_9 9^x$

k. $4^{\log_4 7}$

l. $\left(\log_{27} \frac{1}{3}\right)^2$

m. $5\log 10000$

n. $\log 0.1 + \log 0.01$

o. $\log\sqrt{10} + \log\sqrt[3]{10}$

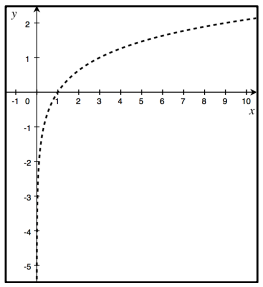
p. $\ln e^7$

q. $e^{\ln\sqrt{e}}$

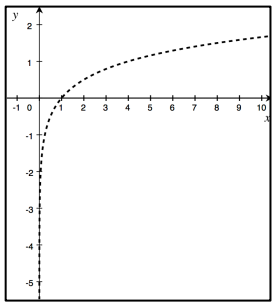
r. $\frac{3\log 10}{2\ln e}$

2. Sketch the transformation of the given log curve, plotting 3 "nice" points.

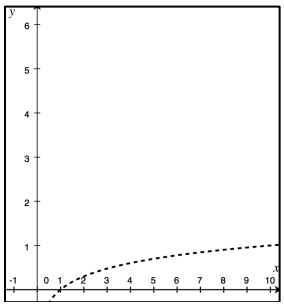
a. $y = \log_3 x - 3$



b. $y = \log_4 (x-2) + 1$



c. $y = 4 - 2\log x$



3. Show necessary work to solve these problems.

a. For a sound with intensity I (in watts per square meter), the loudness $L(I)$ of the sound in decibels is given by the function $L(I) = 10\log\left(\frac{I}{I_0}\right)$ where I_0 is the intensity of a barely audible sound. A recording artist turns up the volume of a track so that the intensity of the sound doubles. Write a function (using $I_0 = 1$) that describes the increase in loudness. Graph the relationship and describe what the graph tells you.

b. The amount of time in months is takes for a club's membership to grow from 100 to any number N is given by $t = \frac{\log N - 2}{\log 1.08}$. Show a graph of this function up to 200 members. Find the difference in time it takes for the membership to go from (first) 110 to 120 and (second) 150 to 160. Explain what this says.

c. A cruise ship stocks milk and obviously cannot restocked while sailing. The number of gallons onboard a 4-day cruise is given by the function $g(t) = 2600 - 1325\ln(t+1)$, where t is measured in days. i. Graph the function. ii. What percentage of available milk was used during the first two days of the cruise? If the ship is delayed by two days getting into port, how much milk did it use the last day?