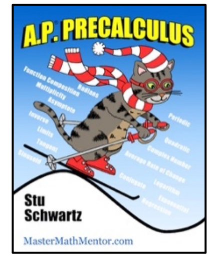
Topic 2.7 – Properties of Logarithms – Classwork



As stated before, there is a LOG key on your calculator as well as a LN key. The LOG key gives the logarithm to base 10 and the LN key gives the logarithm to base *e*. However, there is no dedicated key to find the logarithm of a number to a general base *b*. For instance, if you want to graph the equation of $y = \log_5 x$, there is no one key to accomplish this. However we have *change-of-base* formulas to the rescue.

> Let *a, b,* and *x* be positive real numbers and *a* and *b* are not equal to 1. We can change $\log_a x$ to a different base using any of these three formulas:
>
> | base 10 | base *e* | base *b* |
> |---|---|---|
> | $\log_a x = \dfrac{\log x}{\log a}$ | $\log_a x = \dfrac{\ln x}{\ln a}$ | $\log_a x = \dfrac{\log_b x}{\log_b a}$ |
>
> Most of the time, you will use either of the first two because you prefer to do a calculation using the calculator.

Example 1) Calculate the following (4 decimal places) using your calculator.

a) $y = \log_3 20$ b) $y = \log_8 0.4$ c) $y = \log_{1/4} 7$

So convince yourself that graphing $y = \log_5 x$, use YI = log(X1)//log(5).

Operations With Logarithms

There are three basic rules for operations with logarithms that you must know. These are expressed with common logs but these rules work with logs to any base or the ln function.

> If *a* and *b* are positive numbers, the following properties are true.
>
> 1. $\log(a \cdot b) = \log a + \log b$ 2. $\log\left(\dfrac{a}{b}\right) = \log a - \log b$ 3. $\log a^b = b \log a$

Let's show why each are true:

1.
$$m = \log a \Rightarrow a = 10^m, n = \log b \Rightarrow b = 10^n$$
$$ab = \left(10^m\right)\left(10^n\right) = 10^{m+n}$$
$$\log(a \cdot b) = m + n$$
$$\log(a \cdot b) = \log a + \log b$$

2.
$$m = \log a \Rightarrow a = 10^m, n = \log b \Rightarrow b = 10^n$$
$$a/b = 10^m/10^n = 10^{m-n}$$
$$\log(a/b) = m - n$$
$$\log(a/b) = \log a - \log b$$

3.
$$m = \log a \Rightarrow 10^m = a$$
$$10^{mb} = a^b$$
$$\log a^b = mb$$
$$\log a^b = b \log a$$

Be careful with these rules. It is common for students to mistakenly rewrite $\log x + \log y = \log(x+y)$. This is incorrect as the log function is not distributive. Another common mistake is to equate $\log x - \log y$ to $\dfrac{\log x}{\log y}$ rather than $\log\left(\dfrac{x}{y}\right)$.

Example 2) Use the operation rules to find the values of the following expressions: Confirm with calculator.

a) $\log 25 + \log 4$ b) $\log_2 \dfrac{3}{2} + \log_2 \dfrac{32}{3}$ c) $\log_2 40 - \log_2 5$ d) $\log_5 100 - \log_5 4$

e) $\log 10^{35}$ f) $\log_2 \left(\sqrt{2}\right)^5$ g) $\log 50 + \log 4 + \log 5$ h) $\log_{\frac{1}{3}} \dfrac{4}{3} - \log_{\frac{1}{3}} 12$

Rewriting Logarithmic Expressions

We can rewrite algebraic expressions with our log operation rules to form equivalent expressions. We can take expressions involving complicated processes and write them in a simpler way.

Example 3) Use operation rules to simplify these expressions.

a) $\log\left(3x^4y^2\right)$ b) $\log_2 \sqrt{x^2+4}$ c) $\ln\left(\dfrac{x^3-1}{x^4}\right)$ d) $\log \sqrt[3]{\dfrac{x}{3y^2}}$

We can reverse the process and condense log expressions.

Example 4) Use operation rules to condense these expressions.

a) $2\log x + \log 3$ b) $\dfrac{1}{5}\left(\log_3 x - \log_3 4\right)$ c) $2\ln(x+5) - 3\ln x$ d) $\dfrac{1}{2}\left[\log_6(x+1) - \log_6 x^2\right]$

Solving logarithmic equations

Solving logarithmic equations (statement with logs in them) is simply a matter of writing them exponentially and solving via conventional methods. The rules you just studied can be quite helpful and even necessary as well. Remember when you solve to check for extraneous solutions as the base cannot be negative and the expression for which you are taking the log cannot be 0 or negative as well.

Example 5) Solve the following equations for *x*.

a) $\log_5(2x+5)=2$

b) $\log_3(4x-7)=4$

c) $\log_x(2x+8)=2$

d) $\log_4(x^2-x+2)=\frac{1}{2}$

e) $\log(x-1)+\log 4=2$

f) $\log_6(35x+6)-\log_6 x=2$

g) $\ln(2x-3)+\ln(x+4)=\ln(2x^2+15)$

h) $\log_2(x^2-6x)=3+\log_2(1-x)$

i) $\log(x+5)-\log x=\log(x+1)$

j) $\ln(x-1)-\ln x=1$

Solving Exponential Equations Revisited

We solved exponential equations when it was possible to get a common base on both sides. $2^x = 8$ could be solved by expressing the equation as $2^x = 2^3$ and $x = 3$. $3^x = 12$ provides a different challenge.

Exponential equations are in the form of $a^{f(x)} = b$ or $a^{f(x)} = b^{g(x)}$. To solve such equations, the technique is to take the log (any base but 10 is best) of both sides. And use the fact that $\log a^b = b \log a$.

6) Solve for x algebraically and then 3 decimal places. Verify using the calculator, numerically or graphically.

a) $3^x = 12$

b) $3.4^{x+4} = 29.6$

c) $10000 = 4000(1.06)^x$

d) $3^x = 12^{x-2}$

e) $e^{-x} = 10$

f) $2e^{2x-5} = 24$

g) $3^{3x-4} = 4^{2x-3}$

h) $\left(1 + \dfrac{0.03}{12}\right)^{12t} = 4$

i) $e^{2x} - 5e^x = 6$

j) $\dfrac{5}{1 + e^{-x}} = 2$

Exponential Curve Fitting Revisited *

We have looked at finding an exponential curve passing through 2 points in a previous chapter before our introduction to logarithms. For instance, finding an exponential in the form of $y = a \cdot b^x$ that passes through (0, 3) and (4, 6) had us plugging in the points to that equation.

$$(0,3): 3 = a\left(b^0\right) \Rightarrow a = 3 \qquad y = 3b^x : (4,6) \Rightarrow 6 = 3b^4 \Rightarrow b = \sqrt[4]{2} \qquad y = 3\left(\sqrt[4]{2}\right)^x \text{ or } y = 3(1.1892)^x$$

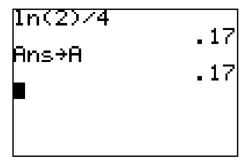
Your calculator has a dedicated key to raise any number to any power. However, further study into calculus will show that this process uses logarithms. So we will now do this problem using logarithms and natural logs, both of which have their uses. Our answers will be either $y = a \cdot b^x$ or $y = Ce^{ax}$. In either case, the constant in front of the exponential is the y-intercept, which in this case is 3. We will usually be given that information.

Example 7) Find the equation of the exponential curve passing through the point (4, 6) with y-intercept = 3.

a) Using $y = a \cdot b^x$ with $a = 3$

b) Using $y = Ce^x$ with $C = 3$

When you calculate the value of a on your calculator, be sure to store it. The calculator's accuracy setting doesn't matter as long as you store the value. Here, I stored it as A. On your function screen, use Y1 = 3e^(AX). On your home screen, confirm this is correct by asking for Y1(4). You should get 6. Y1(8) is therefore 12.



8) Find the equation of the exponential curve passing through the points (5, 5) and (7,13).

9) The O'Hara's live on an island that got devastated by a hurricane and there was no immediate help. They had 15 gallons of drinking water when the hurricane hit and a day later, they had 12 gallons of water. Determine how long the water would last at a constant rate and if they chose to conserve water and have it decay exponentially, how much would be left after a week?

Topic 2.7 – Properties of Logarithms – Homework

1. Use the change of base formula using base 10 to calculate the following to 3 decimal places.

a. $\log_6 50$ b. $\log_2 25$ c. $\log_{12} 24$ d. $\log_4 3$

2. Use the operation rules to find the values of the following expressions: Confirm with calculator.

a. $\log 125 + \log 8$ b. $\log_4 \frac{2}{5} + \log_4 \frac{5}{128}$ c. $\log_5 200 - \log_5 8$ d. $\log_3 \left(\sqrt{3}\right)^7$

3. Given that log 2 = x, log 3 = y and log 7 = z, find the following in terms of x, y and z.

a. log 18 b. $\log 700$ c. $\log 4.2$ d. $\log\left(\sqrt{\frac{7}{12}}\right)$

4. Write the following in terms of logs or natural log (ln) involving expressions of x, y and z.

a. $2\log x + 3\log y$ b. $\log\sqrt{x} - \log y - \log z$ c. $\log x - 1$ d. $\frac{1}{4}\ln x + \frac{2}{3}\ln y - 5\ln z$

e. $2 + 2\log z$ f. $-\log x - \log y - \log z$ g. $\ln y - 2$ h. $\log x\sqrt{\frac{\sqrt{x}}{y}}$

5. Solve each equation in terms of x.

a. $\log_3(2x-2)=2$

b. $\log_7(7x-6)=2$

c. $2\log(0.5x)=6$

d. $\ln\sqrt{x+4}=1$

e. $\ln 3x+\ln 3=3$

f. $\log_5(x+3)-\log_5 x=2$

g. $\log_2(x-1)+\log_2(x+3)=5$

h. $\log(4x+22)-\log(2x+1)=1$

i. $\ln(x+2)-\ln x=-1$

j. $\log(x+6)-\log(x+2)=\log(x+1)$

6. Solve for x algebraically and then 3 decimal places. Verify using the calculator, either graphically or numerically.

a. $5^x = 10$

b. $6^{x-2} = 15$

c. $e^{5x+2} = 30$

d. $5\left(10^{0.5x-1}\right) = 15$

e. $5^{2x-5} = 20$

f. $6^x = 4^{x+3}$

g. $50^{2x+5} = 2^{3x-1}$

h. $\left(e^{2x+1}\right)^2 = 100$

i. $5^{2x} - 3\left(5^x\right) = 18$

j. $\dfrac{500}{5 - e^{-x/2}} = 250$

7. For each problem, you are given two points P and Q, and one coordinate of another point R. You are to i. determine the equation of the exponential curve that passes through P and Q and ii. find the 2nd coordinate of R. Verify using your calculator. (Answers accurate to 3 decimal places)

a. $P\ (0, 3)$ $Q\ (5, 7)$ $R\ (8, \ \ )$

b. $P\ (0, 8)$ $Q\ (12, 15)$ $R\ (7, \ \ )$

c. $P\ (0, 25)$ $Q\ (6, 21)$ $R\ (13, \ \ )$

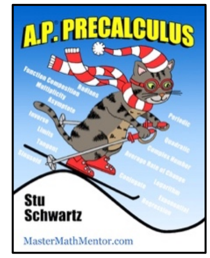
d. $P\ (0, 28)$ $Q\ (11, 25)$ $R\ (9, \ \ )$

e. $P\ (10, 4)$ $Q\ (15, 20)$ $R\ (18, \ \ )$

f. $P\ (-3, 80)$ $Q\ (3, 4)$ $R\ (0, \ \ )$

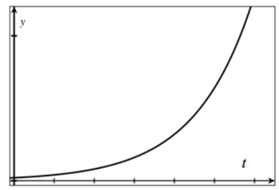
g. A health care worker finds that on the 3rd day of a 10-day cruise, 18 people have the norovirus, which is quite prevalent on cruise ships. Two days later, 42 people have norovirus. Determining trends with only two pieces of data is irresponsible but the captain wants a prediction of how many people will have the illness by the end of the cruise. i. The health care worker uses a straight-line analysis as well as an exponential model. What is the prediction? ii. What is the prediction of how many people came onto the ship with norovirus? iii. Explain why the models are far apart at the end of the cruise but close to each other on day 4.

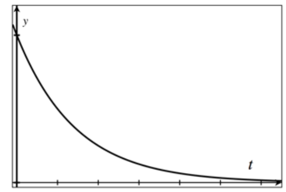
Topic 2.8 – Exponentials Models * – Classwork



Exponential Growth and Decay

When things grow exponentially, they "explode", meaning that as time grows, the number of them increase slowly at first and then faster and faster. When we have exponential decay, the number of them decreases quickly at first and then slower and slower. Here are the graphs of exponential growth and decay as we have seen: All exponential growth and decay curves take this form.





With exponential growth, the more of any substance you have, the faster it grows. With exponential decay, the less of a substance you have, the slower it reduces.

Let's give some examples of real-life applications of exponential growth and decay.

Biology: The number of microorganisms in a culture will increase exponentially until the food supply is exhausted. One organism splits into two, who then split into 4, who split to form 8, and so on. A virus (COVID, AIDS or smallpox for example) will spread exponentially if there is no immunization available. We lived through this stage with COVID.

Population Growth: Populations spread exponentially. The population of the United States is exponentially increasing at the rate of 1.5% a year meaning that its population doubles in about 50 years. The more people there are, the faster the population grows unless something dramatic changes that. COVID had that effect.

Economics and Finance: Companies may show growth in profit that grows exponentially for a while. But usually, this growth then levels off or decreases. We will see examples how bank interest grows exponentially. Retirement funds can decay exponentially until theoretically, there is no more money left in it.

Computer Technology: The processing speed of computers has grown exponentially.

Medicine: When you take a medication, it enters your bloodstream and is absorbed by the body. It gets to work in helping you to feel better. However, with time, the amount of medication in the bloodstream decreases exponentially. Finding the correct dose for you to begin with is usually based on your weight and age.

Also cancer is one of the worst diseases in the world. Cancers cells divide exponentially if not treated.

Archeology: The amount of a substance introduced into our bodies will decay exponentially. Theoretically, there always will be traces of it within the body. This process is used in archaeology to determine how old an artifact is through a process called carbon dating.

Food: When we keep cooked or uncooked food at room or warm temperatures, it gets spoiled. And when spoilage happens (we see green discoloration), the rest of the food spoils at an exponential rate.

Invasive Species: One of the world's most invasive weed, the water Hyacinth, grows exponentially and can clog rivers and block sunlight and oxygen to organisms in the water.

Fire: Damage caused in a fire has an exponential relationship with the duration of burning.

Commodities: The sale of certain objects like smartphones is so rapid as to be seen as exponential.

In real life, exponential growth rarely continues forever. If a disease grows exponentially through a population, the growth will stop when everyone gets the disease. If fire is left untreated, then everything is destroyed and the fire stops as there is nothing left to burn. There are other equations that model such growth but for now, we will concentrate on exponential growth within a limited domain.

Exponential growth/decay can be modeled by the equations below. You are responsible for these equations.

Exponential Growth

$$P = P_0\left(1+r\right)^{t/n}$$

Exponential Decay

$$P = P_0\left(1-r\right)^{t/n}$$

where: P_0 = the initial population (the amount at time zero)
Populations do not have to be people. We call anything that can be counted as a population.

r = the rate of change, usually given as a percent per unit time (a minute, a second, an hour, a year, etc.)

- if a population is increasing at 10% a day, then $r = 0.1$
- if a population is decreasing at 50% a week, then $r = 0.5$
- if a population is doubling every year, then $r = 100\% = 1$
- if a population is tripling every decade, then $r = 200\% = 2$

n = number of time periods it takes for the rate r to occur

- if the population is decreasing by 0.1% every minute, then $n = 1$
- if the population is doubling every two days, then $n = 2$
- if the population is increasing by 10% every half year, $n = 1/2$

t = time, measured in seconds, minutes - the x variable when using the calculator

P = the new population - it will be your Y when using the calculator

Exponential growth or decay modeling usually takes a certain form. You are told about a population (which can represent people, organisms, money, amount of some substance or any countable entity). Usually the size of the population at the start of the problem situation is given.

Information about the growth or decay of the population is given, usually as a rate of change in percentage form and the amount of time it takes for that change to take place. So typically, you are given P_0, the initial population, r and n, From there you write the new size of the population P as a function of time t as shown in the two equations above. To use the calculator as you will most probably have to change t to x and change P to y. Decimal accuracy is dictated by the problem. With money, we usually use 2 decimal place accuracy. With populations of people, we usually round to the nearest integer.

Questions based on the problem situation are usually of two types: 1) finding the size of the population after a certain time duration and 2) finding how much time has to elapse before the population is a certain amount. Both problems can be done mathematically using logs (where only a scientific calculator is needed for arithmetic) or graphically (where a graphing calculator is needed). Students must be able to do the problems mathematically and show the work. Typically, you will do the problems mathematically and then confirm the results graphically.

1) For Dr. Expo, a dentist, the cost of a complete teeth cleaning increases by 3% a year. Its current cost is $125.

a) Write an equation that describes the cost and graph it over 10 years.

b) Find the projected cost for teeth cleaning in 5 years.

c) How long will it take for a teeth cleaning to cost $200? Show mathematically. Confirm graphically.

d) Dr. Linear also charges $125 but increases his price by $5 yearly. When will his cost be less expensive?

2) Assume that the number of bacteria in a bacterial culture doubles every hour and that there are 1,000 present initially.

Variables:

a) Write the equation that describes this growth pattern.

b) Draw a graph that shows the number of bacteria present during the first 12 hours.

c) How many bacteria are present after 8 hours and 15 minutes?

d) Determine mathematically when the number of bacteria will be 1 million. Verify graphically.

3) In # 2), suppose that the number of bacteria triples *every 2 hours.* How does that change the problem?

Variables:

a) Write the equation that describes this growth pattern.

b) Draw a graph that compares the number of bacteria present during the first 12 hours with that of #2.

c) How many bacteria are present after 8 hours and 15 minutes?

d) Determine mathematically when the number of bacteria will be 1 million. Verify graphically.

4) The population of a town is 100,000 and increasing at the rate of 2.15% a year.

Variables:

a) Write the equation that describes this growth pattern.

b) Draw a graph showing the population of this town during the first 50 years.

c) How many people are present after 10 years?

d) Determine mathematically when the population will have doubled. Verify graphically.

5) The population of Allentown, Pa. was 5,000 in the year 1900 and has steadily grown by 12.68% every 4 years since.

Variables:

a) Write the equation that describes this growth pattern.

b) Draw a graph that shows the population of this town during the 20th century.

c) How many people lived in Allentown in the year 2008?

d) Determine mathematically when the population will reach 175,000.

6) The population of a small, western town in the year 1860 was 7,500 and decreasing at the rate of 3.85% every two years.

Variables:

a) Write the equation that describes this decay pattern.

b) Draw a graph that shows the population from 1860 to the year 2020.

c) How many people lived in the town in the year 1935?

d) How many people live there today (2023)?

e) In what year was the population cut in half? Do mathematically and verify graphically.

Exponential Decay Problems

Half-life is the time required for a quantity to reduce to half its initial value. The term is commonly used in nuclear physics to describe how quickly unstable atoms undergo, or how long stable atoms survive, radioactive decay. The term is also used more generally to characterize any type of exponential or non-exponential decay. For example, medicine refers to the biological half-life of drugs and other chemicals in the human body.

Carbon-14 is a radioactive isotope that decays over time. It is present in all life, whether it is living or dead. It decays very slowly. It takes 5,730 years for the amount of carbon-14 in a human body to reduce by 50%. We say the half-life of carbon-14 is 5,730 years.

Measuring the amount of Carbon-14 in a sample from a dead plant or animal such as a piece of wood or a fragment of bone provides information that can be used to calculate when the animal or plant died. The older a sample is, the less Carbon-14 there is to be detected. This technique called *carbon-dating* allows archeologists to determine the approximate age of an object.

7) Use the fact that the half-life of carbon-14 is 5,730 years and that when someone dies, the percentage of carbon-14 in the body is at 100%.

a. Write the equation that describes the percentage of carbon-14 in the body as a function of time.

b. Draw a graph that shows the percentage of carbon-14 left after 10,000 years.

c. What percent of carbon-14 will be remaining after 1,000 years?

d. How many years will it take to have only 10% of carbon-14 left?

e. If an artifact from the Trojan War (1100 BC) were found today, what percentage of carbon-14 would be remaining? (Answer for the year 2023).

8) Not all half-lives are that long. Some elements have relatively small half-lives. In the following table are some basic elements and their half-lives. If you started with 1,000 grams of the element, determine how much would remain in two minutes. Show your formula.

Element	**Half-life**	**Remaining after 2 minutes**
Neon-23	37.24 seconds	
Oxygen-10	26.91 seconds	
Beryllium-11	13.81 seconds	
Nitrogen-16	7.13 seconds	
Carbon-15	2.5 seconds	

9) In the problem above, determine how long it takes for Neon-23 to reduce to 1 gram.

10) Using the technique we just developed in # 9, do the same for the other elements in # 8.

Element	**Half-life**	**Tim to reduce to 1 gram**
Oxygen-10	26.91 seconds	
Beryllium-11	13.81 seconds	
Nitrogen-16	7.13 seconds	
Carbon-15	2.5 seconds	

Exponential Growth in the Financial Field

One of the basic uses of exponential growth is that of money and investments. Money gains interest while sitting in a bank and people taking out loans have to pay interest on those loans because if the money were still in the bank, it would be gaining interest.

There are two types of interest, simple and compound interest. Simple interest is exactly that, you put a sum of money in the bank (principal) and gain the same amount of interest every year. No banks give simple interest. Sometimes, parents give a loan to a child and asked to be paid back with simple interest, which is doing that child a great favor. If a parent loaned his child $5,000 to buy a car and charged him 4% interest over 5 years, the amount the child will pay back is according to the formula:

$\boxed{P = P_0 + P_0 rt}$ where P_0 is the principal, r is the rate of interest and t is the time in years.

The amount that the child will pay back is $5000 + 5000(0.04)(10) = 5000 + 2000 = \$7,000$. Since the loan was $5,000, the child paid $2,000 in interest.

Compound interest is the method that all banks use to give interest. Compound interest means interest upon interest. For example, assume we put $5,000 in the bank at 4% interest compounded annually. Let's determine the amount of interest for each year.

Principal (P_0)	Interest in year: ($P_0 r$)	New Principal
5,000.00	1: 200.00	5,200.00
5,200.00	2: 208.00	5,408.00
5,408.00	3: 216.32	5,624.32
5,624.32	4: 224.97	5,849.29
5,649.29	5: 233.97	6,083.26
6,083.26	6: 243.33	6,326.59
6,326.59	7: 253.06	6,579.65
6,579.65	8: 263.19	6,842.84
6842.84	9: 273.71	7,116.55
7,116.55	10: 284.66	7,401.21

Note how the interest increases each year. Also note that the final principal is much more than the $7,000 we had with simple interest.

When banks give compound interest, they compound it using different time periods. Compounding interest quarterly, for example, means that they give interest 4 times a year. Suppose $5,000 is invested in a bank that gives 4% annual interest, compounded quarterly. How much will the investor have at the end of a year?

Realize that that though the bank gives 4% interest a year, it will give $\frac{4\%}{4} = 1\%$ interest every quarter of a year. Complete the chart, determining the amount of interest the money earns each quarter and the principal at the end of each quarter.

Quarter	Principal	Interest	New Principal
1			
2			
3			
4			
5			
6			
7			
8			

You can see that compounding quarterly give an extra $3.03 in that first year more than compounding annually did. In the second year, compounding quarterly gave $3.26 more.

Rather than completing this chart for longer periods of times, we use the exponential growth formula to find the worth of an investment at time *t*. Recall this formula is:

$$\boxed{P = P_0\left(1+r\right)^{t/n}}$$

if P_0 is the initial principal, r is the rate of interest, and n is the amount of time for the interest rate to be paid.

However, since r is typically the annual rate of interest (APR) and compounding gives interest over shorter periods of time, realize that the rate of interest is now $\frac{r}{n}$ where n is the number of times per year that interest is compounded. If t is the number of years that interest is given, then the number of compounding periods will equal (nt). The basic formula for exponential growth can be used, but with n usually fractional, it is far easier to tweak the formula to the one below. This will be further explained at the end of this section but for now, let's just use the formula.

Formula for compound interest: $P = P_0\left(1+\frac{r}{n}\right)^{nt}$

where: P_0 = initial principal (principal at time t = zero)
r = Annual Percentage Rate (APR) – this is what banks publish – your rate of interest yearly
t = time in years
n = number of compounding periods – banks give interest in varying number of periods
- bi-annually – interest every two years (n = 0.5) – quite rare
- annually – interest once a year (n = 1) – somewhat rare
- semi-annually - interest twice a year (n = 2) – somewhat rare
- quarterly – interest every 3 months (n = 4) – somewhat common
- monthly - interest every month (n = 12) – some banks do it
- weekly – interest every week (n = 52) – quite rare
- daily – interest every day (n = 365) – most banks do it

11) If I invested $5,000 in a bank giving 4% APR interest for 10 years, how much would I have according to the following compounding methods? Set your calculators to 2-decimal places and show your formula.

Bi - annually	**Annually**	**Semi - annually**
Quarterly	**Monthly**	**Weekly**
Daily	**Hourly**	**Every Minute**

We can see that the parent who loaned their child $5,000 in simple interest did the child a great favor.

Also note that although the amount increases, it increases by smaller and smaller amounts. Increasing from compounding weekly to compounding daily give an extra 98 cents over 10 years. Compounding daily to every minute gives an extra 16 cents over 10 years. So, by increasing the number of compounding periods, we find that there is no pot of gold. Increasing the number of compounding period helps, but ultimately very little.

Continuous Compounding

Note that as *n* gets larger, there is essentially no difference in the amount of interest you will gain. Some banks compound continuously. That means that rather than a time period passing and interest accumulating, money is always gaining interest. In any split second, the amount is infinitesimally small, but it all adds up. The formula for continuous compounding is

$$P = P_0 e^{rt}$$

where P_0 is the initial principal, *r* is the annual rate of interest and *t* is time in years. Here is our first real-life use of the number *e* which you recall is 2.718281828… You have an *e* button on your calculator - in fact you have two of them. The first is above the division sign and the second is above the ln key which allows for taking *e* to a power. Continuous compounding is the best interest method possible for an investor.

As stated before, the number *e* comes up in unexpected places in mathematics. How compounding an infinite number of times is related to *e* will be answered in a calculus course.

12) How much is $5,000 invested at 4% APR compounded continuously worth?

13) $100,000 is invested for 20 years at 3.5% interest APR. How much more money will be gained by continuous compounding over daily compounding?

14) Suppose you invested $1 in a bank at 5% interest and simply left it there to gain interest. If it compounded interest according to the continuous compounding formula, how much would be in the bank after…

1 year	10 years	50 years	100 years	200 years	250 years	500 years

So, could you put $1 in the bank and write it in your will that your 20th descendant will get the money? You can but it should be noted that after a period of time (differs from state to state), you must make contact with the bank in regard to this account. If there is no contact, the money is given to the state. Also, realize that based on inflation, that money will not buy what it does today.

15) You plan to invest $10,000 in a CD (certificate of deposit) for 5 years. There are four CD's available to you with the following rates and compounding. Which one give you the best deal? Show your work.

a) 3.15% - annually   b) 3.1% - quarterly   c) 3.05% - monthly   d) 3.03% - continuously

16) At the birth of their child, Mr. and Mrs. DeGenerous wish to give him a new car (worth $20,000) upon his graduation from high school when he is 18 years old. How much should they invest at 3.25% APR compounded quarterly in order to have the money available to this boy upon graduation? How about if they can find a bank that gives the same interest compounded continuously?

17) George Dubya wants to double his money in 10 years. Find the rate of interest he would need if his money compounds continuously. Find the rate if he is satisfied tripling his money in 20 years.

18) The rule of 72 states that the length of time it takes an investment to double can be approximated by dividing 72 by the APR (where 8% is 8). Compare the actual length of time it takes for money to double compared to the rule of 72 for different interest rates compounded continuously by completing the table.

19) A local "Money Superstore" advertises that you can borrow $100 for 14 days and pay "just" a $13 finance charge. What annual interest rate are you paying if money compounds continuously? If you borrowed the same amount for a year, what would you be paying back?

20) In the last problem, the payback appears outrageous but is not quite as bad as it looks The fine print says that the $13 fee is divided into 10% of the loan plus a one-time $3 verification fee. Repeat the problem calculating r and then determine the payback for the loan at the end of one year.

We have two formulas for exponential growth and decay: $P = P_0\left(1 \pm r\right)^{t/n}$ and $P = P_0\left(1 \pm \frac{r}{n}\right)^{nt}$. How do we know which one to use? In reality, there is only one: $P = P_0\left(1 \pm r\right)^{t/n}$. However, in the case of financial growth where principal is compounded quarterly or daily, etc., the rate r is an annual rate. Since it is compounded n times, the rate is thus $r\left(\frac{1}{n}\right) = \frac{r}{n}$. We give interest once every $\frac{1}{n}$ years so $\frac{t}{1/n} = nt$. Hence our formula becomes $P = P_0\left(1 \pm \frac{r}{n}\right)^{nt}$. However, it is easy to get confused over what n represents so it is better to have two formulas: one for populations and the other for compounding (when you are given an annual rate of growth or decay, but the compounding is over smaller units of time).

Also, don't get hung up over formulas but understand what the formula actually does. In exponential growth or decay, you are interested in only three pieces of information: the amount you start with, percentage you have after each cycle, and how many cycles. In simple cases, you don't need the complicated formula.

21) I borrow $1,000 from a loan shark charging 10% interest/day. How much must I pay back in 10 days?

22) Suppose we have 500 grams of a substance that is decaying exponentially. The rate of decay is 1.9% a) annually, b) quarterly, c) daily. Determine how much of the substance is left in a year. Show the use of the formula and the simplification of the formula.

23) In #22, the rate of decay is 1.9% continuously. Determine how much of the substance is left in a year.

Topic 2.8 – Exponentials Models – Homework

1. Assume that the number of bacteria in a bacterial culture triples every hour and that there are 20 present initially.

   Variables:

   a. Write the equation that describes this growth pattern.

   b. Draw a graph that shows the number of bacteria present during the first 12 hours.

   c. How many bacteria are present after 6 hours and 45 minutes?

   d. Determine mathematically when the number of bacteria will be 1 million. Verify graphically.

2. For # 1, suppose that the number of bacteria doubles every 3 hours. How does that change the problem?

   Variables:

   a. Write the equation that describes this growth pattern.

   b. Draw a graph that shows the number of bacteria present during the first 12 hrs.

   c. How many bacteria are present after 6 hours and 45 minutes?

   d. Determine mathematically when the number of bacteria will be 1 million.

3. The population of a city is 1,000,000 and increasing at the rate of 1.97% a year.

   Variables:

   a. Write the equation that describes this growth pattern.

   b. Draw a graph that shows the population of this town during the first 60 years.

   c. How many people are present after 10 years?

   d. Determine mathematically when the population will have doubled.

.

4. The population of New York, NY was 7.9 million in the year 1970 and has averaged 3.1 % growth every 10 years since.

   Variables:

   a. Write the equation that describes this growth pattern.

   b. Draw a graph that shows the population of this town for the next 60 years.

   c. How many people lived in New York in the year 2008?

   d. Determine mathematically when the population will reach 10 million.

5. At Area Six High School with population 1,024, 18 students have colds on September 1, and the number of students who have or had colds that school year doubles every 11 days.

   Variables:

   a. Write the equation that describes this growth pattern.

   b. Draw a graph that shows the number of students *not having had a cold* over four months.

   c. How many students did not have a cold after 30 days?

   d. When was half the student body affected?

6. The population of Russia in 2006 was 142 million and because of declining births and life expectancy the growth rate is decreasing at 0.6% per year.

   Variables:

   a. Write the equation that describes this decay pattern.

   b. Draw a graph that shows the population from 2006 to 2050.

   c. How many people will live in Russia in 2075?

   d. In what year will the population be down to 100 million? Do mathematically. Verify graphically.

7. The half-life of marijuana varies from person to person but is normally between 1 to 10 days. Assume that the half-life is 5 days for a regular marijuana user. Suppose such a person uses marijuana today (you don't know how much – so deal with it in terms of percent). Today he has 100% of the marijuana in his bloodstream.

a. Write the equation that describes this decay pattern.

b. Draw a graph that shows the percent of marijuana over a month.

c. How much marijuana will be remaining after a week?

d. How many days will it take for less than 1% of the original marijuana to be left in that person's body?

e. Rick is to be tested on day 2 for marijuana in the bloodstream but gets the test delayed for 24 hours. What percent of the original marijuana did he lose in that time?

8. The half-life of Cobalt-60 is 5.27 years. How long will it take for 75% of the material to decay?

9. Suppose a material such that 10 years ago there were 1500 grams and today, there are 1300 grams. What is the material's half-life?

10. A rare manuscript is thought to go back to the time of the ancient Greek - 800 BC. Since it is made from parchment from trees, carbon dating with carbon 14 (half-life 5730 years) is performed. It is found that the manuscript contains 38% of the carbon in living trees. Is it possible that this manuscript dates back that far? Show the analysis that leads to your conclusion.

11. A sum of money is invested in a bank at various rates, for various times, and compounded in different methods. Complete the table and find the new principal. Show your formula.

Principal	Rate	Time	Method	New principal
a. $500	6%	5 years	Annually	
b. $2,000	3.5%	6 years	Quarterly	
c. $25,000	6.2%	20 years	Monthly	
d. $4,000	7.3%	8 years	Daily	
e. $850	3.3%	2.5 years	Weekly	
f. $2,400	5.8%	9 months	Continuously	
g. $100,000	4.5%	15 years	Bi-annually	
h. $100,000	4.5%	15 years	Annually	
i. $100,000	4.5%	15 years	Semi-annually	
j. $100,000	4.5%	15 years	Quarterly	
k. $100,000	4.5%	15 years	Monthly	
l. $100,000	4.5%	15 years	Daily	
m. $100,000	4.5%	15 years	Hourly	
n. $100,000	4.5%	15 years	Continuously	

12. Mr. and Mrs. B. Prepared decide at age 25 when they get married that they want to have $100,000 in the bank at the time of their retirement at age 65. They put away much of their wedding money into the bank at 5.5% interest. How much must they put in the bank, compounded daily, to have $100,000 when they retire?

13. Complete the chart to find the amount of time necessary for P dollars to double if interest is compounded continuously at the following rates.

r	3%	4%	5%	6%	7%
t					

14. Complete the chart to find the amount of time necessary for P dollars to triple if interest is compounded continuously at the following rates.

r	3%	4%	5%	6%	7%
t					

15. The total interest I paid on a home mortgage of P dollars at interest rate r for t years is given by the formula:

$I = P\left[\dfrac{rt}{1-\left(\dfrac{1}{1+r/12}\right)^{12t}} - 1\right]$. Suppose your mortgage of \$400,000 had to be paid off over 30 years at 5.5%.

a. What is the total interest and amount to be paid back over 30 years.

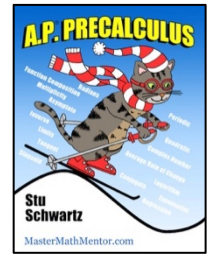
b. Suppose you decided to pay it back over 15 years. How do those figures change?

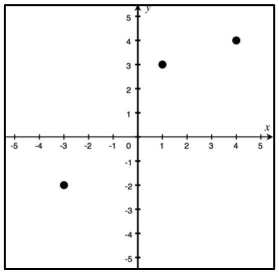
c. Suppose you were able to get an interest rate of 5%. How does the 30-year payback figures change?

16. A maker of pretzels, potato chips, and popcorn decides to lower the amount of salt it uses in these products by 15%. However, the company is concerned that too quick a change will cause them to lose sales. So, they decide to reduce the salt content by 2.5% quarterly over the period of 1.5 years. A typical 10-ounce bag contains 1,750 milligrams of salt. How much salt will this bag contain after 1.5 years? Did the company achieve their goal of reducing the salt by 15%? If not, by what percentage was the salt reduced?

17. Suppose the company decides to reduce the salt in their products by 25%, but again, make the change over time: 2.5% quarterly. How long will this take? Show your work.

Topic 2.9 – Regression – Classwork

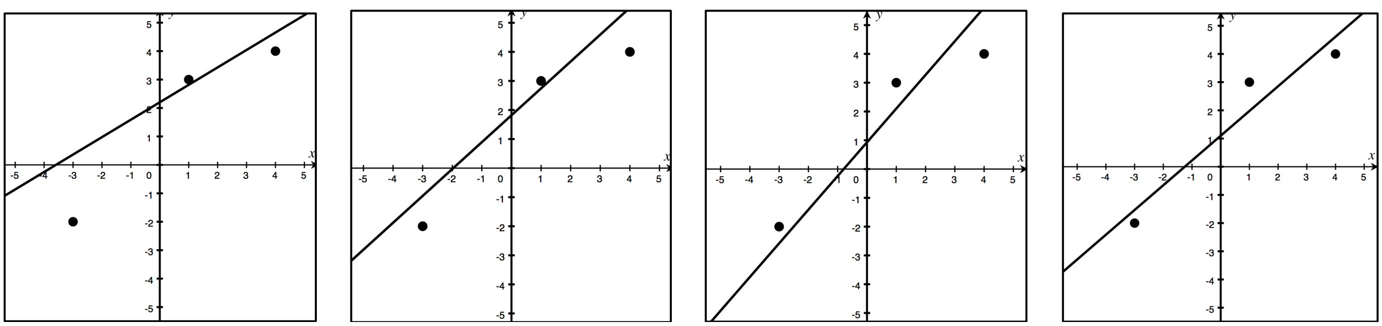




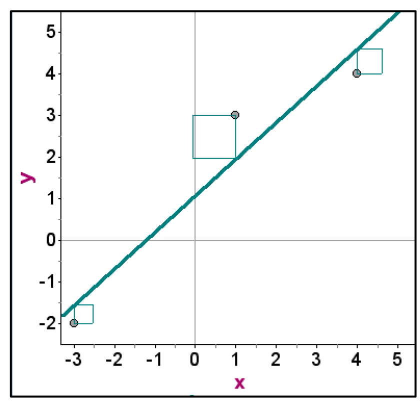
Let's do a little review on the behavior of lines. We know that given any two points, there is one unique line that passes through them. But how about if we have 3 points?

It is clear that there is no line that satisfies these 3 points to the left as they are not collinear. The slopes between any pair of points are not the same.

So we are interested in the line that does the best job of describing three points. What would that line look like? It cannot pass through all three points but should be close to all of them. That line of best fit is not easy to determine but it is clear that as we look at the 4 lines below, this line of best fit approximation is better the farther to the right we go.



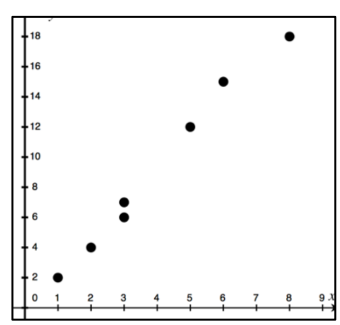
The method to find this line of best fit involves minimizing the distances from the points to this hypothetical line. Since some points are below the line and some are above the line, we square this distance to make everything positive. The line of best fit minimizes the sum of the areas of these squares for all possible lines. For that reason, this line as shown to the right is called the *least-squares regression line*.



The process of finding this least-square regression line (LSRL) is called *linear regression*. The technique to find it mathematically is taught in AP statistics and is not required for AP precalculus. However, your calculator is capable of performing linear regression and that is where our focus will be.

Let us perform linear regression with the points (1, 2), (2, 4), (3, 6), (3, 7), (5, 12), (6, 15), (8, 18). The first thing that should hit you is that these points do not create a function because there are 2 values for $x = 3$. If x represents the minutes basketball players played in a game and y represents the number of points the player scored, it is possible and reasonable to have two points (3, 6) and (3, 7).

Scatterplots – show the relationship between two variables. The values of one variable appears on the horizontal (x) axis and the other variable appears on the vertical (y) axis. Each individual appears as a point in the plot. When doing algebra, we just called this process plotting points but when working with data, we call it creating a scatterplot.

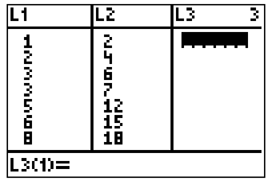


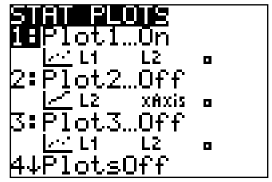
Modeling: When we model with linear regression, we are taking a dataset derived from a real-life application and attempting to generate the least-square line that best fits the data. We use that equation to predict a value of y based on a value of x.

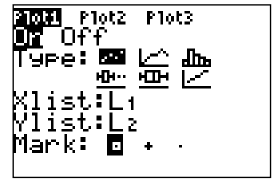
Using the Calculator

Place the data in your lists, usually L1 and L2: STAT | EDIT

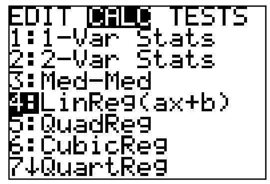
Set up your plots: 2nd | Y =
Then: ZOOM | 9 : ZoomStat Be sure that you have no graphs in your Y = list.
Your scatterplot should appear.







STAT | CALC | 4 : LinReg(ax + b)



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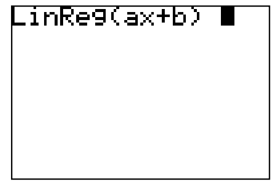
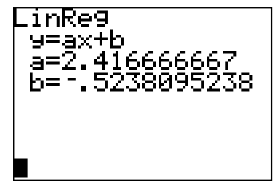


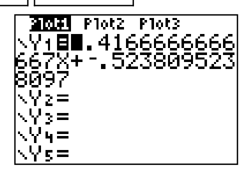
If the lists aren't specified, the calculator automatically use L1 and L2 for regression.
If you want regression on other lists, you must specify (ex: LinReg(ax+b) L2,L4)
If you want to actually graph the line, do these steps:

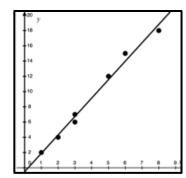


STAT | CALC | 4 : LinReg(ax + b) | VARS | Y - VARS | Function | Y1 | Enter

Press Y = and your equation will be in Y1:

Press Graph and you will see the line.





Meaning of Slope in LSRL: The slope of the LSRL represents the change in y for two consecutive x values. In this case, the slope of 2.417 means that as x changes from n to $n+1$, the y-value changes by 2.417.

Your line of best fit (the regression line) can be used to predict the value of y for a given x, For instance, if you want the predicted value of y when $x = 4$, use Y1(4) and you get 9.1430. If you want the predicted value of y when $x = 25$, use Y1(25) = 69.8937.

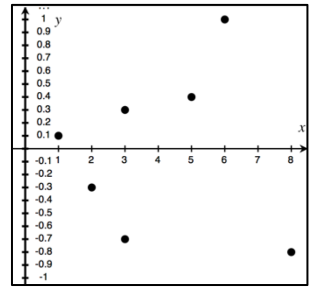
Predicting with the LSRL: When you use the line of best fit to predict y for x within the given data, it is called *interpolation*. In the example above, we are interpolating to predict the value of y when $x = 4$.

When you use the line of best fit to predict y for x outside the given data, it is called *extrapolation*. In the example above, we are extrapolating to predict the value of y when $x = 25$. While interpolation is fairly safe, the further from the data set you are extrapolating, the riskier it is. In this case $x = 25$ is far from the largest x-value of 8 and reliability for the prediction of $y(25)$ is open to question.

Residual: A residual is the difference between an observed value of the response variable and the value predicted by the LSRL. The formula for a residual is: $\boxed{\text{Residual} = \text{observed } y - \text{predicted } y}$

In our data set (we have the point (5, 12). The predicted value for $x = 5$ is $y(5) = 11.56$. So the residual for $x = 5$ is $11.56 - 12 = -0.44$. Residuals can be positive if the observed value is greater than the predicted value and negative is the observed value is less than the predicted value. We calculate the *error* in our LSRL for any value of x as $|\text{Residual}|$.

For any LSRL, the sum of the residuals is always zero.



To the right is a *residual plot*. It is a graph of all the residuals. Some are positive, some are negative. Although you will probably not be asked to draw it (although there is a way to do it on the calculator), it may be given to you. The haphazard appearance of the points is a good indication that a linear fit to the data is appropriate. Later we will see residual plots that have a pattern to them, indicating that the data is best fit by a curve, either quadratic or exponential.

1) Last Christmas season, Walmart conducted a study as to the amount of time in checkout lanes its customers had to wait. On Saturdays and Sundays of its holiday season, it opened a different number of checkout lanes for customers between 1 PM and 4 PM, its busiest times. The measurement was the average wait time for a customer to go through the lane and complete the transaction. A different number of lanes were opened each day. The data is below

Date	11/22	11/23	11/29	11/30	12/6	12/7	12/14	12/15	12/21	12/22	12/29	12/30
Lanes	5	12	11	7	12	8	6	10	8	6	4	8
Avg Wait Time (Min)	12.2	4.2	4.4	6.75	3.8	5.75	10.4	6.5	6.25	9.2	1.1	5.6
Predicted Wait Time												
Residual												

a) Find the LSRL. b) Explain the role of the slope of the LSRL. c) Complete the table. d) Find the predicted wait time if 9, 20 and 25 lanes were open. Explain your confidence in each. e) If Walmart decides that it wants its customers to wait no longer than 2 minutes, how many lanes should it open?

2) A large airport tracks the number of delays for flights starting in the 7 AM hour, 8 AM hour, through 9 PM hour where t is the number of hours past 7 AM. a) Find the LSRL. b) Explain the role of the slope of the LSRL. c) Complete the table. d) Find the predicted delays between 1:30 PM and 2:30 PM as well as 12 midnight – 1 AM. Explain your confidence in each. d) Between what times would you expect 16 delays?

Time	7 AM	8	9	10	11	12 PM	1	2	3	4	5	6	7	8	9
Delays	2	2	4	6	10	9	10	15	17	20	21	28	25	24	28
Predicted Delays															
Residual															

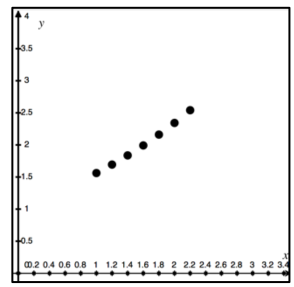
Exponential Regression

Sometimes, it is not clear whether linear regression is appropriate for a dataset. While the LSRL can always be performed, using it for predictions is worthless and detrimental if it is not the appropriate model. Here is a review of linear and exponential growth.

> A variable y grows *linearly* if it adds (or subtracts) a fixed increment over equal x-increments.
> That is: if $y_{n+1} - y_n$ is pretty much the same, that is evidence that the data is fits a linear model.
>
> A variable y grows *exponentially* if it multiplies (or divides) a fixed ratio over equal x-increments.
> That is: if $\frac{y_{n+1}}{y_n}$ is pretty much the same, that is evidence that the data is fits an exponential model.

Let's use this example. Suppose we have the dataset:

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	1.562	1.694	1.837	1.992	2.161	2.343	2.541



We create the scatterplot and it appears that a straight line is appropriate for this data. Your eye might detect a slight curve but surely it cannot make a difference. Or can it?

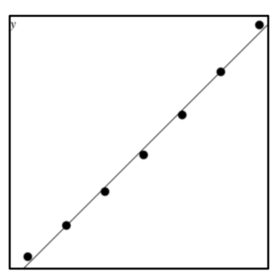
However, let us look at the change in y-values Δy between the given x-values which have equal 0.2 increments.

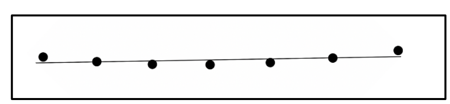
x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	1.562	1.694	1.837	1.992	2.161	2.343	2.541
Δy	0.132	0.143	0.155	0.169	0.182	0.198	

Notice how the difference between y-values are not only different, they are growing. That is a signal that the linear model is not appropriate.

So, let's look at the ratio of y-terms - $\frac{y_{n+1}}{y_n}$. We are looking at $\frac{1.694}{1.562} = 1.085$, $\frac{1.837}{1.694} = 1.094$, $\frac{1.992}{1.837} = 1.094$, …
From the first few of these calculations, you can see that the ratio is staying pretty much the same. This is a signal that the exponential model is appropriate.

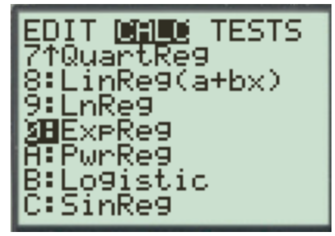
We can verify this with a residual plot although this takes a bit of work. But there is a simple way to get a sense of it. Draw the least-squares regression line as shown to the right and tilt your head as to make the line horizontal. This is a residual plot and you can see that there is a pattern to the residuals – starting positive, becoming negative and then positive again.

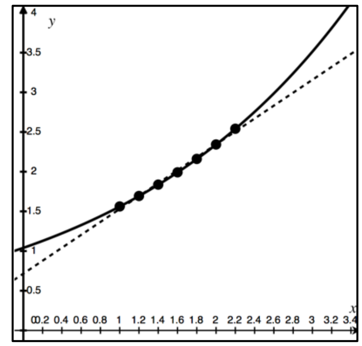




The calculator has the ability to find the best exponential model (the curve that best fits the points) just as it did for linear regression and we will use it. In AP statistics, we learn the mathematics behind it. We take the log of the y data, do linear regression with x vs. log y, generate the LSRL and then work back to find the exponential model $y = 10^{\text{LSRL}}$. In AP precalculus, we will just take advantage of the built-in exponential regression.

STAT | CALC | 0:ExpReg | VARS | Y–VARS | Y1 will generate the Regression screen, perform the exponential regression in the form of $y = ab^x$. In this case the exponential regression result is $y = 1.041(1.5)^x$.





Let us view both the results of linear regression and exponential regression on one graph. The question of which model is appropriate is better answered in AP statistics. But it should be clear that for us, the appropriateness question is based on how we will use the models. If we wish to interpolate, for instance find the predicted value of *y* at $x = 1.5$, there is little to choose between the two models. Linear regression gives 1.937 and exponential regression gives 1.912, a 1.3% difference.

But if we wish to extrapolate and find the predicted value of *y* at $x = 3.5$, the model makes a large difference. Linear regression gives 3.565 and exponential regression gives 4.303, a 20.7% difference. Just looking at the graph tells you that the larger the value of *x*, the spread between the two models increases exponentially.

In AP precalculus, you are responsible for doing both types of regression, predicting values for both models, and defending which is the better model by a residual plot (which might be given to you). In many cases, the data will show a clear curve and linear regression will then be off the table. That will leave us with a decision between an exponential model and a quadratic model, examined next.

3) Petroleum has become the most single source of energy for developed nations in this and the last century. In recent years it has become the cause of economic dislocation and war. The table below shows the growth of annual world crude oil production measured in millions of barrels per year (mb). Data is given only through 1970 when a Mideast war touched off a dramatic price increase and a change in the previous pattern of production.

Year	1880	1890	1900	1910	1920	1930	1940	1950	1960	1970
Oil (mb)	30	77	149	328	689	1412	2150	3803	7674	16690

a) Enter the data into your calculator with *x* being the number of years after 1880. What is the least-squares regression line? Use it to predict the oil production in 1955.

b) Give a reason why the linear regression model is not appropriate.

c) Perform exponential regression and use it to predict the oil production in 1955.

d) If the pattern had continued, what would both models predict for 1980?

4) The reselling cost of a car gets smaller with every passing year because of depreciation. Below is a table that shows the percentage value of a new BMW and a new Ford's original cost based on mileage.

Miles (thousands)	0	10	20	30	40	50	75	100	125	150
BMW (%)	100	78	60	45	36	28	19	13	9	7
Ford (%)	100	70	49	36	28	25	15	10	8	6

a) Explain why linear regression is not appropriate for either vehicle.

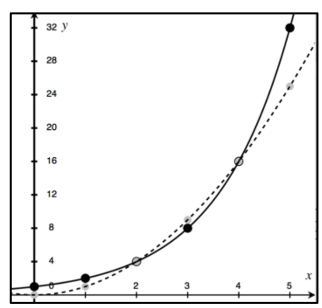
b) Create exponential models for each vehicle and use them to find the value of an $80,000 BMW and a $35,000 Ford after 25,000 miles.

c) Predict the number of miles when each vehicle will be worth half of its original value?

Quadratic Regression

If we are given data and wish to model it, we first consider linear regression. We look at a residual plot and can tell at a glance if we can rule out linear regression. If there is a pattern to the residuals, then we need a curve to model the data, not a line. But what type of curve? There are many types. In this course, we just consider exponentials and quadratics.

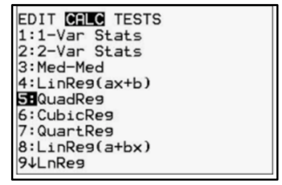
Let's consider two simple models to the right … one exponential: $y = 2^x$ (the solid curve) and the other quadratic: $y = x^2$ (the dashed curve). The shapes of these curves, at least with these points are very similar and it would be quite difficult to label one as exponential and the other as quadratic by looking at them. And if we were to perform interpolation within these points, there wouldn't be much difference. At $x = 2.5$, the exponential gives us 5.627 and the quadratic gives us 6.25. But as usual, if we are extrapolating, we would like to be sure of our model.



AP statistics will teach techniques to determine which model is superior but if the data is given with equal increment x-values, there is a relatively simple method to determine this information. Let us look at the ratio from one term to the the previous one: $\frac{y_{n+1}}{y_n}$.

x	0	1	2	3	4	5
$y=2^x$	1	2	4	8	16	32
$\frac{y_{n+1}}{y_n}$	2	2	2	2	2	
$y=x^2$	0	1	4	9	16	25
$\frac{y_{n+1}}{y_n}$		4	2.25	1.78	1.56	

As we see to the left, in a exponential, the ratio between consecutive terms remains constant. But in a quadratic, the ratio between consecutive terms either increases or decreases. In this case it decreases. Of course, it is rare that our models are perfect so exponential ratios won't be exactly the same from point to point and quadratic won't always increase or decrease. But these are trends to examine. If we are not sure, we do both regressions and interpolate or extrapolate both.



Performing quadratic regression is similar to linear and exponential. We enter data in L1 and L2 and press: STAT | CALC | 5:QuadReg | VARS | Y–VARS | Y1. The calculular gives an answer in the form of $y = ax^2 + bx + c$ and places it in Y1.

5) The U.S. Department of Health and Human Services characterizes adult males as obese if they meet the weight criterion for their height.

Height	4'10"	5'0"	5'2"	5'4"	5'6"	5'8"	5'10"	6'0"	6'2"	6'4"
Weight (lbs)	143	153	164	174	186	197	209	221	233	246
Ratio of terms										

a) Complete the ratio of terms row. Is there evidence that the data better fits a quadratic than an exponential?

b) Perform both exponential and quadratic regression and use the models to determine the obese weight to the nearest pound for someone 5'9" as well as 6'10". Comment on your confidence of these numbers.

6) A shot putter throws 8 times in a competition. Sensors determine the height of his throw. The table below show the distance of his throws and the height, both in feet. We wish to predict height from the distance thrown.

Distance (ft)	32	34	35	30	34.5	32	39	35
Height (ft)	13.8	12.6	11.8	14.9	11.2	12.9	7.6	11.0

a) Explain why quadratic regression is the best of the 3 methods. Find the equation.

b) Which of his throws has the greatest error in the quadratic model?

Logarithmic Curve Fitting

Since we fit exponential curves with two points we should also be able to determine a logarithmic curve with two points as well. Exponential curves are concave up while logarithmic curves are concave down. The general problem is to find a logarithmic curve passing through a point with a given *x*- intercept in the form of $y = a + b\log x$. It should be pointed out that we can create this logarthmic function with any base but for now, let's use natural logs. This method can get messy as we will see.

7) Find an equation of the exponential passing through the 2 points and find the *y*-value for the given *x*-value.

a) (1, 0), (5,2) $x = 15$

b) (3,0), (6, 9) $x = 5$

This technique is built into the calculator as well. Please your points in L1 and L2 and use te keystrokes. STAT CALC 9:LnReg VARS Y–VARS Y1. The equation is placed in Y1.

8. A student with no previous typing experience takes a three-week typing course and after each day takes a test. The test requires the student to type for 3 minutes with 98% accuracy. The speed is recorded in words per minute (wpm). Here is the data. Day 0 refers to a pre-test taken before the course started. When you enter it into your calculator, use 0.1 instead of 0 as 0 is not in the domain for logarithmic regression.

Day	0	1	3	6	7	10	12	14	16	18	19	21
Speed (wpm)	10	20	33	39	43	49	56	59	60	62	64	66

a) Enter the data and perform both quadratic and logarithmic regression. Explain why exponential regression is not possible.

b) Predict the speed in words per minute for both models on day 5 and day 35.

c) Based on answer b and the graphs of each, which is the best model?

Topic 2.9 – Regression – Homework

1. Does washing your hands lead to fewer colds? A study was done to help decide this question. People chosen to be part of the study were asked to keep track of the number of colds/flu they had over a period of one year. At the end of the year, they were given a questionnaire and one of the questions asked them to estimate the number of times they wash their hands a day. This did not include showers or baths. For the people washing their hands only once, twice, three times, etc. a day, the average number of colds was calculated. The results are in the table below. (2-decimal place accuracy)

# of times per day washing hands	0	1	2	3	4	5	6	7	8	9	10	16
Average number of colds	5.52	5.71	5.54	4.85	5.54	4.93	4.03	3.72	2.18	2.12	1.5	1.62

   a. Create a scatterplot and find the LSRL.

   b. Explain the meaning of the slope in the LSRL.

   c. What is the largest residual?

2. A diver is investigating a wreck under the water and has to come up to the surface slowly to prevent him betting the bends. Following is a table detailing his depth from the time he starts ascending. Since he is under water, his depth is shown as negative numbers.

time (min:sec)	0 sec	0.30	1:00	1:40	2:20	3.00	3.30	4:40	5:30	6.00	6:30
depth (ft)	–240	–225	–203	–189	–185	–164	–155	–146	–130	–125	–100
Predicted depth											
Residual											

   a. Create a scatterplot and find the LSRL.

   b. Explain the meaning of the slope in the LSRL.

   c. Complete the table by finding the predicted depth to the nearest foot and the residual.

   d. Using the model, predict the diver's depth at these times. Comment on the confidence of your prediction.

      a) 2 minutes, 50 seconds   b) 5 minutes   c) 7 minutes, 10 seconds

   e. Using the linear model, how long would your predict before the diver reaches the surface.

3. Below are the U.S. box office gross in millions of dollars for the first 12 weekends of the movie Avatar, The Way of Water, with the 5th weekend data missing.

Weekend	1	2	3	4	6	7	8	9	10	11	12
Gross (millions)	113.6	78.2	51.0	33.4	12.8	9.6	7.8	4.5	3.7	2.9	2.5

a. Create a scatterplot and a model. Explain why the model is appropriate.

b. What is the predicted gross in week 5?

c. What is the week 1 residual?

d. Studio executives will pull the movie from theatres when the gross is below $1 million. How many weekends is the movie expected to be in the theatres? Solve mathematically.

4. In the very early days of 2020, reports from Wuhan China gave the number of people affected by what was called back then, the new Coronavirus, what we now call COVID-19. There is no corroboration as to the accuracy of the numbers in the table below. Obviously we did not know then what we do now. But the fear was that the growth of the virus was exponential.

Date	Jan 20	Jan 21	Jan 22	Jan 23	Jan 24	Jan 25	Jan 26	Jan 27	Jan 28	Jan 29
Day	1	2	3	4	5	6	7	8	9	10
Cases	278	326	547	639	916	2,000	2,700	4,400	6,000	7,700

a. Create a scatterplot and an exponential model $y = a \cdot b^x$.

b. Explain the meaning of b in the exponential equation.

c. Develop a formula determining how many days it will take k people to have the virus. Complete the table.

d. Suppose only half the day 1 people had the virus but the transmission rate stayed the same. How would that affect the number of days to reach the number of cases in the table on part c?

5. Corey posts a new video to his YouTube channel. The table below shows the number of people who have viewed the video after a given number of days.

Days	1	2	3	4	5	8	12	15
Viewers	1	5	10	19	29	79	184	295

a. What is evidence that the growth of viewership is quadratic rather than exponential?

b. Find and graph each model.

c. Find the error in both models on day 8.

d. Use both models to estimate mathematically when Corey will have 1,000 viewers. Confirm graphically.

6. Angelo's Pizza has 5 sizes of pizza. Rather than offering slices of pizza, they have a pizza called "cookie size". The size and prices of their circular pizzas are in the table below:

Size	**Cookie**	**Personal**	**Small**	**Medium**	**Large**
Diameter (inches)	3	6	10	14	16
Cost	\$4.75	\$7.75	\$11.50	\$14.75	\$17.00

a. Generate a linear regression model, an exponential model, and a quadratic model for diameter vs. cost.

Linear: Exponential: Quadratic:

b. If Angelo's has two new pizza sizes called a "mini" (8-inch diameter) and an "Oh-My-God" (42-inch diameter). How much does each model suggest they should charge for it?

Linear: Exponential: Quadratic:

c. Which model makes the most sense and why?

7. Find an equation of the exponential passing through the 2 points and find the y-value for the given x-value.

a. (1, 0), (12, 20) $x = 5$

b. (2,0), (15, 3) $x = 20$

8. Students were asked to participate in an experiment. They attended several lectures on a topic and at the end, were told they would be given an exam on the subject. They were given various amounts of time to "cram" for the exam. Once they took the exam, their scores were calculated and averaged based on the amount of time they were given to study. The results are in the table below.

Hours	1	3	5	12	24
Score	90	82	74	63	58

a. Generate a quadratic regression model, an exponential model, and a logarithmic model for hours vs score.

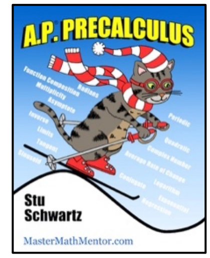
Quadratic: Exponential: Logarithmic:

b. What does each model predict for taking the exam, 23 minutes after the last lecture, 2 hours later, and 36 hours later?

Quadratic: Exponential: Logarithmic:

c. Graph all 3 models and use them to explain why one of the models is clearly better.

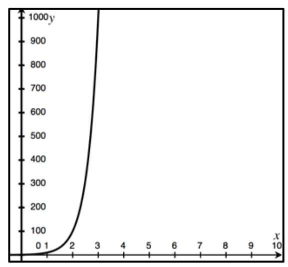
Topic 2.10 – Semi-Log Plots – Classwork

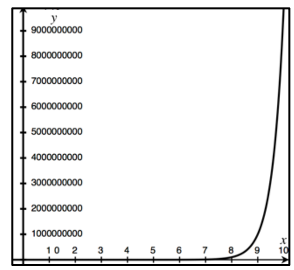


In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings. Even though both caused substantial damage, the earthquake in Japan was 100 times as strong as the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.

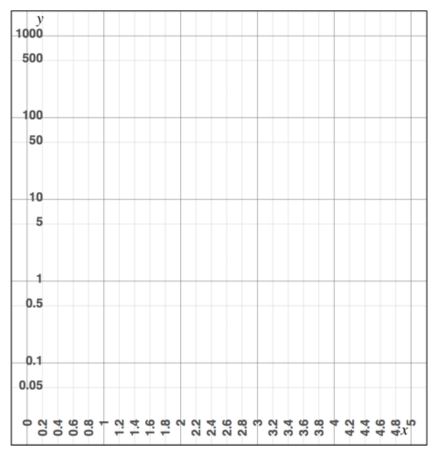
The most common standard of measurement for an earthquake is the *Richter scale*, developed in 1935 by Charles F. Richter of the California Institute of Technology. The Richter scale is used to rate the *magnitude* of an earthquake. This is calculated using information gathered by a seismograph that measures ground movement.

The Richter scale is *logarithmic*, meaning that whole-number jumps indicate a tenfold increase. In this case, the increase is in wave amplitude. That is, the wave amplitude in a level 6 earthquake is 10 times greater than in a level 5 earthquake, and the amplitude increases 100 times between a level 7 earthquake and a level 9 earthquake. Generally, the amount of damage is based on the Richter scale but is also dependent on the composition of ground and the number, design, and placement of manmade structures in the area.



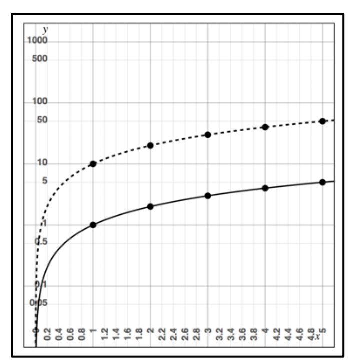


To get a sense of the intensity of an earthquake, we graph $y = 10^x$ where x represents the level of the earthquake as designed by Richter. The strongest earthquake ever recorded was the 9.5 quake that struck Chile in 1060. So we graph $y = 10^x$ over a domain of (0, 10). As you can see, the graph goes off the scale vertically at $x = 3$. So let's adust the y-scale to view all of the graph. That requires the y-scale to have a maximum value of 10^{10} = 10 billion.



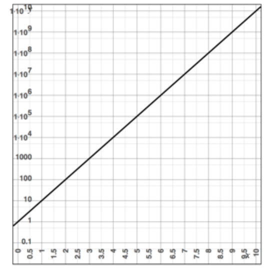
The issue now is that all the detail from $x = 0$ to $x = 8$ is lost. The graph appears to be horizontal which we know is not the case. The graph, while accurate tells us very ltttle.

So we create a semi-logarithmic graph. We make the vertical axis logarithmic (equal scale between powers of 10) while the horizontal axis remains linear (equal space between numbers). There are no negative numbers on the y-axis, since we can only take the logarithms of positive numbers. Note that the halfway point between powers of 10's: 0.5, 1.5, 2.5 are not at the halfway point vertically. That is because, as example, $10^{0.7} \approx 5, 10^{1.7} \approx 50,$ etc,



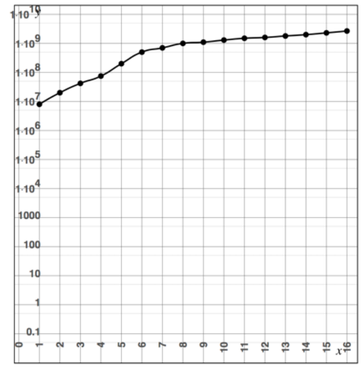
So let's graph the simple function $y = x$ on a semi-log graph. We know this passes through (0, 0), (1, 1), (2, 2), (3,3), etc. On semi-logarithmic axes, the graph of $y = x$ is a curve and not a straight line. It still passes through the points mentioned, with the exception of (0, 0), because we cannot take the log of zero. Also graph is $y = 10x$, passing through (1,10), (2,20), (3,30), etc. So, on semi-logarithm graph paper, linear expressions appear logarithmic. So what is the advantage of such a system?

So let's now graph our earthquake problem, $y = 10^x$ which represents the intensity of an earthquake based on the Richter scale. Our exponential curve now becames a straight line allowing us to clearly see the intensity of any Richter scale earthquake. All the detail that was lost in our curve graphed on the Cartesian axes is gained on the semi-log graph.



However, this graph can be deceiving. An earthquake measuring 6.0 is 10 times more destructive than one measuring 5.0 and a 7.0 quake is 100 times more destructive than a 5.0. The semi-log graph distorts the actual magnitude of the intensity.

1) The figure to the right represents YouTube's monthly active viewers (MAV) on a semi-log graph with $x = 1$ corresponding to 2006. Estimate the following.



a) How many more MAV's did YouTube have in 2021 than in 2006?

b) What is YouTube's average growth rate over those years?

c) Are MAV's growing faster between 2010 and 2011 or 2020 and 2021? Explain.

Your calculator does not draw semi-log graphs. Many graphing software packages do not include it. So what do you when you want to graph an equation that shows huge exponential growth and decay? Many times, the simple answer is to solve for log y and graph that equation being sure to show and state that the vertical axis is no longer y but log y. So if $y = ab^x \Rightarrow \log y = \log\left(ab^x\right) = \log a + \log b^x = \log a + b \log x$.

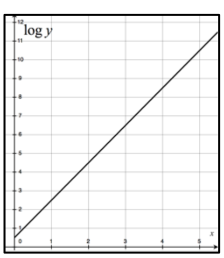
2) Express the exponential graph in logarithmic form so it can easily be graphed on on an x-vs. log y scale.

a) $y = 5.5\left(10^x\right)$

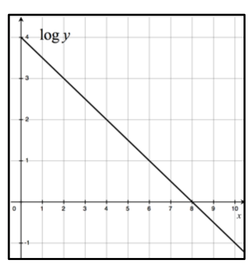
b) $y = 80\left(2^{-x}\right)$

3) Given the graph of log y, generate the exponential equation in the form $y = ab^x$.

a)



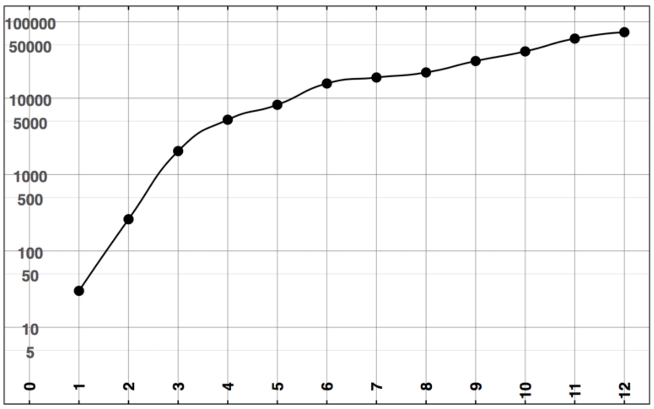
b)



Topic 2.10 – Semi-Log Plots – Homework

1. The swine flu (H1N1) pandemic of 2009 was not nearly as bad as COVID but still packed a punch. About 61 million people worldwide were infected by HIN1 in the first year. The semi-log graph of the data gives the total number of cases every week at the start of the pandemic. To the best of your ability, estimate the total number of cases for the first 12 weeks by completing the table.

Week	1	2	3	4	5	6	7	8	9	10	11	12
Cases												



a. Between what two weeks is the slope of the graph the greatest. Estimate the slope and describe its meaning.

b. Between what two weeks is the slope of the graph the smallest. Estimate the slope and describe its meaning.

c. Explain why your answer in b is greater than your answer in a.

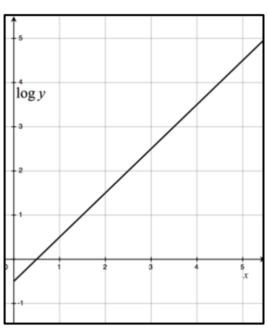
2. Express the exponential graph in logarithmic form so it can easily be graphed on on an x-vs. log y scale.

a. $y = 3.4\left(10^{x/10}\right)$

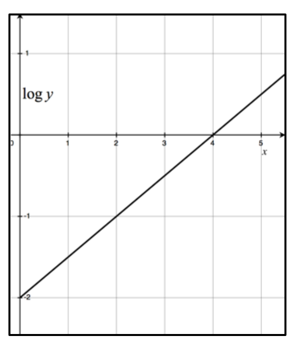
b. $y = \dfrac{9}{100}\left(1.5^{-x}\right)$

3. Given the graph of log y, generate the exponential equation in the form $y = ab^x$.

a.



b.



Unit 3
Exponential and Polar Functions