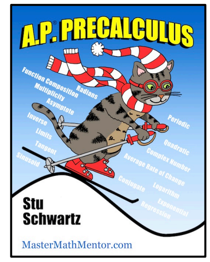
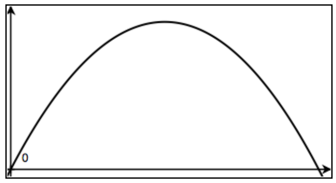
Topic 4.1 – Parametric Functions – Classwork



Until now, we have been representing graphs by single equations involving variables x and y. We will now study problems with which 3 variables are used to represent curves. Consider the path followed by an object that is propelled into the air at an angle of 45º. If the initial velocity of the object is 48 feet per second, the object follows the parabolic path given by

$$y = x - \frac{x^2}{72}$$



However, although you have the path of the object, you do not know *when* the object is at a given time. In order to do this, we introduce a third variable t, called a *parameter*. By writing both x and y as a function of t, you obtain the parametric equations:

$$x = 24t\sqrt{2} \qquad \text{and} \qquad y = -16t^2 + 24t\sqrt{2}$$

From this set of equations, we can determine that at the time $t = 0$, the object is at the point (0, 0). Similarly, at the time $t = 1$, the object is at the point $\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$.

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called *parametric equations* and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a plane curve.

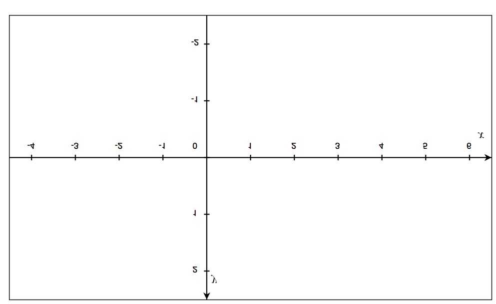
When sketching a curve by hand represented by parametric equations, you use increasing values of t. Thus, the curve will be traced out in a specific direction. This is called the *orientation* of the curve. You use arrows to show the orientation.

We define a *parametric function* $h(t) = (x(t), y(t))$ where x and y are two functions.

Example 1) Sketch the curve described by the parametric equations:

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2} \qquad -2 \le t \le 3$$

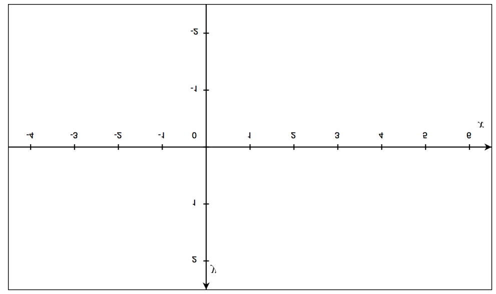
t	-2	-1	0	1	2	3
x						
y						



Example 2) Sketch the curve described by the parametric equations:

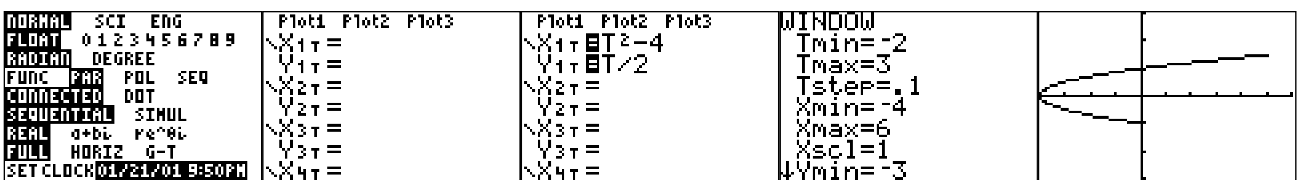
$x = 4t^2\text{-}4$ and $y = t$ $\quad -1 \le t \le 3/2$

t	-1	-0.5	0	0.5	1	1.5
x						
y						

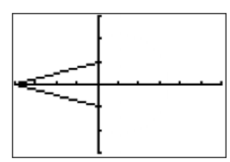


Note that both examples trace out the exact same graph. But the speed is different. Example 2's graph is traced out more rapidly. Thus, in applications, different parametric equations can be used to represent various speeds at which objects travel along paths.

Note that the TI-84 calculator can graph in parametric mode. Go to MODE and switch to Parametric mode as shown below. Your Y= button now gives you the screen below. The X,T,θ,n button now gives a T when pressed. The equation in example 1 can now be generated. You set the T values by going to your WINDOW and placing them in Tmin and Tmax. Xmin, Xmax, Ymin, Ymax work as before Note that the arrow showing orientation does not display when you graph a parametric on the calculator. If you are asked to draw a parametric on an exam, you must include it.



Tstep controls the accuracy and speed of your graph. Large values of Tstep give speed but not little accuracy. For instance, if Tstep of 2 were used, the graph above would appear as the figure on the right. The calculator will determine values of T = –2, 0, and 2 and connect them with lines. Small values of Tstep give a lot of accuracy at the cost of speed.



Finding a rectangular equation that represents the graph of a set of parametric equations is called *eliminating the parameter*. Here is a simple example of eliminating the parameter.

Example 3) Eliminate the parameter in $x = t^2 - 4$ and $y = \dfrac{t}{2}$

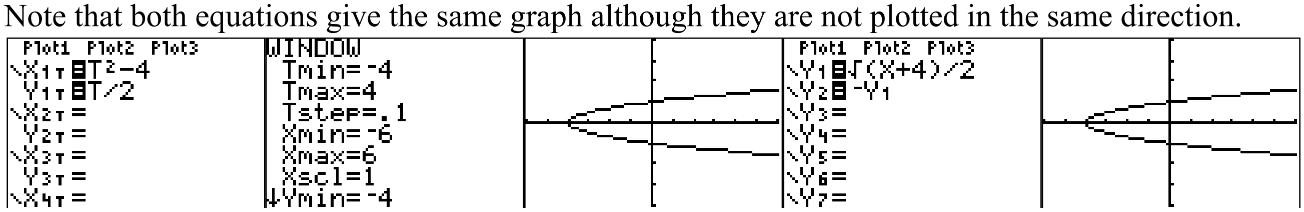
• Solve for t in the second equations

$t = 2y$

• Substitute in the second equations and simplify

$x = 4y^2 - 4 \Rightarrow 4y^2 = x + 4 \Rightarrow y = \dfrac{\pm\sqrt{x+4}}{2}$

Note that both equations give the same graph although they are not plotted in the same direction.



Example 4) Eliminate the parameter in the following parametrics. In each problem, it will usually be easier to solve for t in one equation than the other. Then graph, showing that the 2 equations graph the same curve.

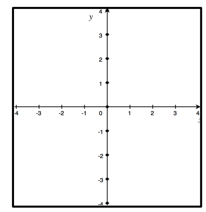
a) $x = t - 3$ and $y = t^2 + \sqrt{t} - 2$

b) $x = 3t + 2$ and $y = \dfrac{1}{2t - 1}$

c) $x = \dfrac{t}{2}$ and $y = \sin(t+1) - 1$

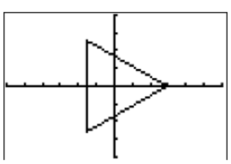
d) $x = \dfrac{1}{\sqrt{t-1}}$ and $y = \dfrac{t}{t-1}$ $t > -1$

Example 5) Sketch the curve represented by $x = 3\cos t$ and $y = 3\sin t$ $0 \le t < 2\pi$

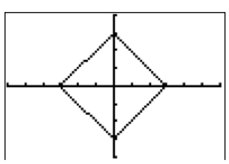


- Solve for $\cos\theta$ and $\sin\theta$ in both equations.
- Use $\sin^2\theta + \cos^2\theta = 1$ to form an equation using only x and y. and multiply by 9 to get.

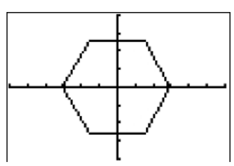
- This is a graph of a circle centered at (0, 0) with diameter endpoints at (3, 0), (–3, 0), (0, 3), and (0, –3). The circle is traced *counterclockwise* as θ goes from 0 to 2π. When Graphed on the calculator in degree mode, different values of Tstep will graph different pictures.



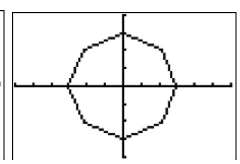
TStep = 120



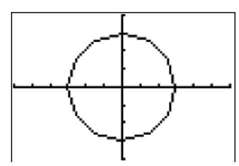
Tstep = 90



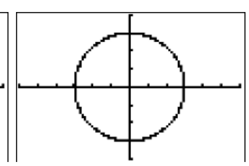
Tstep = 60



Tstep = 45



Tstep = 30



Tstep = 1

Since $x = \cos t$ and $y = \sin t$, $0 \le t < 2\pi$ generates a circle of radius 1, centered at the origin, $x = \cos t + h$ and $y = \sin t + k$, $0 \le t < 2\pi$ generates a circle of radius 1, centered at the point (h, k). And $x = \pm a(\cos t + h)$ and $y = \pm a(\sin t + k)$, $0 \le t < 2\pi$ generates a circle of radius $|a|$, centered at the point (ah, ak). The graph is created in a counterclockwise direction. If the parametric equations are $x = \pm a(\sin t + h)$ and $y = \pm a(\cos t + k)$, $0 \le t < 2\pi$, the circle is created in a clockwise direction.

Using this technique allows us to work backwards. For instance, if we have the circle $x^2 + (y-1)^2 = 25$, we can write that $\cos t = x$ and $\sin t = y - 1$. So our parametric equations are $x = \cos t$ and $y = \sin t + 1$.

However, there is nothing to stop us from saying that $\cos t = y$ and $\sin t = x - 1$. That gives us $x = \sin t + 1$ and $y = \cos t$. The circle in rectangular form just gives us the picture. The parametric form gives us the picture and the orientation. If I asked you to draw a circle, some would draw it clockwise and others counterclockwise. But the result is the same. The parametric form gives us not only the completed graph but how the graph was generated.

Example 6) Determine the parametric equation's center and radius. Confirm by changing to function notation.

a) $x=2(\sin t-4), y=2(\cos t+2)$

b. $x=-\frac{3}{2}\cos t+1, y=\frac{3}{2}\sin t-4$

Example 7) Change the circle with the given center and radius to parametric form.

a) $C(7,-5), r=10$

b) $C\left(\frac{-1}{2},-\frac{1}{3}\right), r=\frac{3}{5}$

Example 8) In general, finding a set of parametric equations for a given function is easy to do. There are infinite ways of doing so. The easiest is to simply let $t=x$ and then replace your y with t. Find parametric equations for the following functions.

a) $y=x^2-2x+3$

b) $y=\sqrt{\frac{2x-4}{3x-1}}$

For the problems above, let, for example, $x=t+2$ and find the resulting parametric equations.

Behavior of Parametric Functions

In recognizing the graphic behavior of a parametric function, we usually are interested in several pieces of information:

• Horizontal Extrema – how far right and left the function goes. This can be determined by identifying the maximum and minimum values of $x(t)$ as well as examining the end behavior of $x(t)$. If $x(t)$ is linear and $-\infty < t < \infty$, then there is no maximum or minimum x value of the curve. If there is a max or min, we are usually interested in what it is and the value of t where it occurs.

• Vertical Extrema – how far up and down the function goes. This can be determined by identifying the maximum and minimum values of $y(t)$ as well as examining the end behavior of $y(t)$. If $y(t)$ is linear and $-\infty < t < \infty$, then there is no maximum or minimum y value of the curve. If there is a max or min, we are usually interested in what it is and the value of t where it occurs.

• Intercepts – where the function crosses the x- and y-axes and the corresponding value of t,

To find the x-intercept of the function, solve $y(t) = 0$ for t. Plug that value of t into $x(t)$.
To find the y-intercept of the function, set $x(t) = 0$ for t. Plug that value of t into $y(t)$.

Example 9) For each of the following parametric functions for $-\infty < t < \infty$, find the maximum and minimum values of both x and y as well as the value of t where it occurs. Also find the x- and y-intercepts as well as the value of t where it occurs. Confirm graphically.

a) $f(t) = (6t + 3, 5 - 10t)$

b) $f(t) = (t^2 + 1, 2t - 3)$

c) $f(t) = \left(4 - \frac{t}{2}, 4 - t^2\right)$

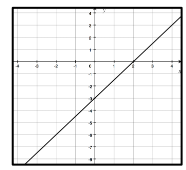
d) $f(t) = (t^2, 8 - t^3)$

When we analyzed functions, we focused on the shape of its graph. When analyzing parametric functions, we are not only interested in the shape but how the shape was created. Since parametric functions model movements of objects, we are interested in analyzing the motion itself rather than the final picture.

Increasing functions: As t increases,
if $x(t)$ increases, the object is moving to the right. If $x(t)$ decreases, the object is moving to the left.
if $y(t)$ increases, the object is moving up. If $y(t)$ decreases, the object is moving down.

We can also compute the average rate of change of $x(t)$ and $y(t)$ between two values of t independently. It makes sense that the ratio of the average rate of change of y to the average rate of change of x between two values of t gives the slope of the parametric graph between those two values of t as long as the average rate of change of x is not zero.

Be careful with this concept. The graph of $f(t)=(4-2t,3-3t)$ is to the right. As t increases, both x and y decrease meaning that the graph is steadily moving down to the left. However, the slope of the line is positive between any two values of t. The slope of the curve has nothing to do with the value of t.



Example 10) For the previous problems, find intervals when an object is moving left, right, up and down by completing the table. Also find the average rate of change of both x and y between $t = 1$ and $t = 3$ as well as the slope of the parametric curve between those two values of t.

a) $f(t)=(6t+3,5-10t)$

b) $f(t)=(t^2+1,2t-3)$

c) $f(t)=\left(4-\frac{t}{2},4-t^2\right)$

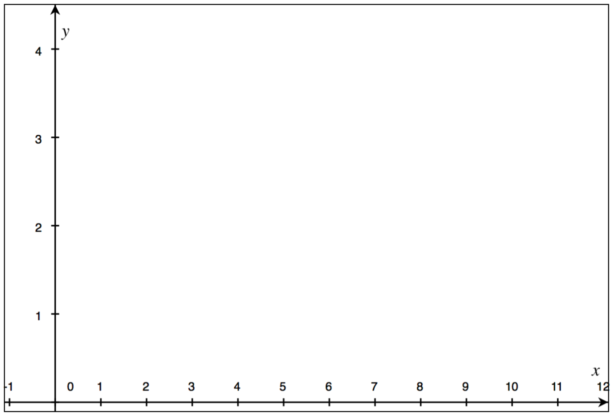
d) $f(t)=(t^2,8-t^3)$

Parametric curves may have loops, cusps, vertical tangents and other peculiar features. Parametric curves are not necessarily functions as they fail the vertical line test. Parametric curves are less about the shape of the curve and more about how that shape comes about.

Example 11) At any time t with $0 \le t \le 10$, the coordinates of P are given by the parametric equations:

$$x = t - 2\sin t \quad \text{and} \quad y = 2 - 2\cos t$$

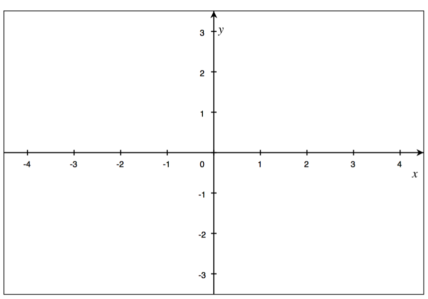
Sketch this using your calculator.



- Points corresonding to integer values of t are shown. At $t = 1$, P has coordinates of $(-0.68, 0.92)$. At this instant, P is heading almost due north.
- The full curve is not a function as some x-values have more than one y-value.
- The graph shows x and y-values but not t-values.
- The bullets on the graph occur at equal time intervals but not at equal distances from each other as P speeds up and slows down as it moves.

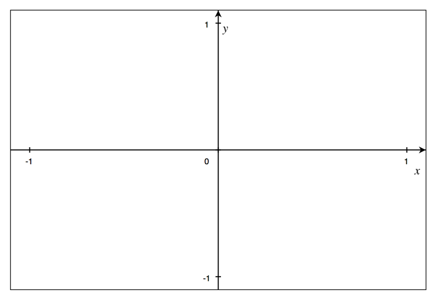
Example 12) Sketch these on your calculators.

a) $x = 2\cos t + 2\cos(4t),\ y = \sin t + \sin(4t) \quad 0 \le t \le 5$

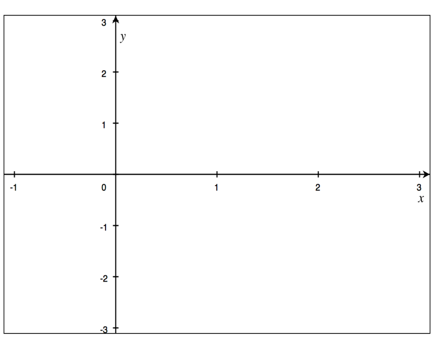


b) $x = \sin(5t),\ y = \sin(6t) \quad 0 \le t \le 2\pi$

Lissajou Curve



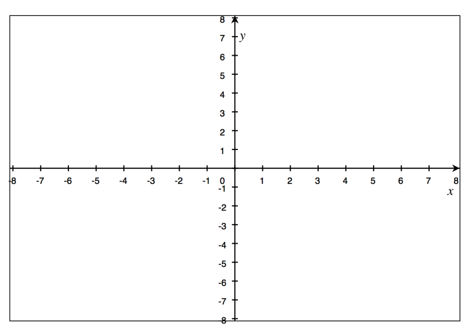
c) $x = \dfrac{3\sin t\cos t}{t},\ y = \dfrac{3\sin^2 t}{t} \quad -6\pi \le t < 6\pi$



Cochleoid

d) $x = \sqrt{t}\cos t,\ y = \sqrt{t}\sin t \quad -20\pi \le t < 20\pi$

d.



Fermat's Spiral

Projectile Motion *

If a projectile is launched at a height of h feet above the ground at an angle of θ (measured in degrees or radians) with the horizontal. If the initial velocity is v_0 feet per second, the path of the projectile is modeled by the parametric equations:

$$x = v_0 t\cos\theta \qquad y = h + v_0 t\sin\theta - 16t^2$$

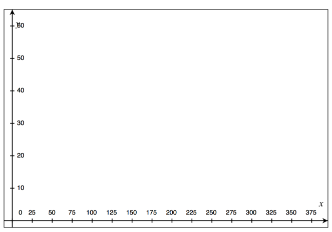
The $-16t^2$ component is the effect of gravity. This only applies to the *y*-coordinate.

Example 13) For each problem, use the calculator to graph two parametric equations for a projectile fired at the given angle at the given initial speed at ground level. Then use the calculator's trace ability to estimate the maximum height of the object as well as its range.

a. $\theta = 30°, v_0 = 90$ ft/sec
$\theta = 30°, v_0 = 120$ ft/sec

Par. Eq 1:

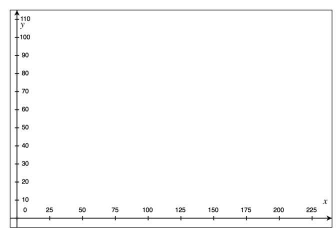
Par. Eq 2:



b. $\theta = 30°, v_0 = 90$ ft/sec
$\theta = 70°, v_0 = 90$ ft/sec

Par Eq 1:

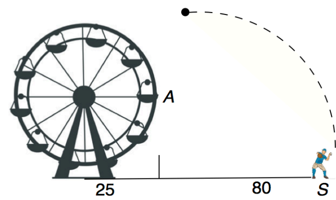
Par Eq 2:



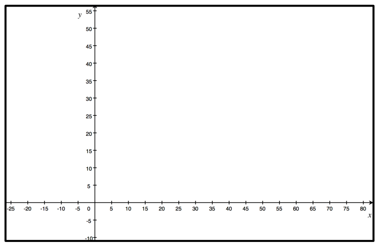
When these are graphed, typically there are two questions asked: How high does the object get and what is its range (how far horizontally does the object travel? In parametric mode on the TI-84, there is no MAX function as there was in function mode. You need to use the TRACE command to plot the particle's location at various values of *t* to see where the *y*-value hits its maximum. To find its range, again use the trace function to see where the *y*-value equals zero. More accuracy is gained by making Tstep smaller.

Example 14) A baseball player is at bat and makes contact with the ball at a height of 3 ft. The ball leaves the bat at 110 miles per hour towards the center field fence, 425 feet away which is 12 feet high. If the ball leaves the bat at the following angles of elevation and no wind, determine whether or not the ball will be a home run with a) $\theta = 17°$ and b) $\theta = 18°$ and c) $\theta = 18°$. Show your equations and explain your answers.

Example 15) Sheldon is standing on the ground at point *S*, a distance of 80 feet from the bottom of a Ferris wheel that has a 25-foot radius as shown in the figure to the right. His arm is at the same height as the bottom of the Ferris wheel, 5 feet off the ground. Andrea is on the Ferris wheel which makes one revolution counterclockwise every 20 seconds. At the instant she is at point *A*, Sheldon throws a ball to her at 56 ft/sec at an angle of $60°$ to the horizontal.



a) Create a sketch of the problem situation on numbered axes.



b) Generate parametric functions that describe the path of the Ferris wheel.

c) Generate parametric functions that describe the path of the ball.

d) Using the TRACE key, does the path of the ball and the path of Andrea appear to intersect at the same time?

e) How far is Andrea from the ball at $t = 2.45$ seconds?

f) Use your grapher to find the time when Andrea and the ball are closest.

Topic 4.1 – Parametric Functions – Homework

1. Consider the parametric equations $x = \sqrt{t}$ and $y = 2t - 1$

   a) Complete the table

t	0	1	2	3	4
x					
y					

   b) Plot the points (x, y) in the table and sketch a graph of the parametric equations. Indicate the orientation of the graph.

   c) Find the rectangular equation by eliminating the parameter.

2. Consider the parametric equations $x = 4\cos^2\theta$ and $y = 2\sin\theta$

   a) Complete the table

θ	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
x					
y					

   b) Plot the points (x, y) in the table and sketch a graph of the parametric equations. Indicate the orientation of the graph.

   c) Find the rectangular equation by eliminating the parameter.

3. In the following exercises, eliminate the parameter and confirm graphically that the rectangular equations yield the same graph as the parametrics. Be sure you take domain and range of the parametric into account.

   a. $x = 4t - 1$ and $y = 2t + 3$

   b. $x = t - 3$ and $y = t^2$

   c. $x = \sqrt[3]{t}$ and $y = 3 - t^2$

   d. $x = t^2 - 1$ and $y = t^2 + t$

e. $x = t - 2$ and $y = \dfrac{t}{t-2}$

f. $x = |t-3|$ and $y = t + 3$

g. $x = \sec^2\theta$ and $y = \tan^2\theta$

h. $x = \cos\theta$ and $y = 4\sin\theta$

i. $x = e^t$ and $y = e^{-t}$

j. $x = t^5$ and $y = 5\ln t$

4. For each rectangular equation, find 2 sets of parametrics, the first by letting $x = t$ and the second by setting x equal to the given expression and finding the y-component.

a. $y = 2x^2 - 3x - 2, \quad x = t - 1$

b. $y = \dfrac{2x-5}{x^2 - x - 2}, \quad x = t + 2$

5. For each of the following parametric functions for $-\infty < t < \infty$, find the maximum and minimum values of both x and y as well as the value of t where it occurs. Also find the x- and y-intercepts as well as the value of t where it occurs. Confirm graphically.

a. $f(t) = (2 - 8t, 4t - 4)$

b. $f(t) = (2t - t^2, 2 - t)$

c. $f(t) = (|t-2|, |t+1|)$

d. $f(t) = (t^2 - 1, t^4 - 4t)$

6. For the previous parametric functions, find intervals when an object is moving left, right, up and down by completing the table. Also find the average rate of change of both x and y between $t = -1$ and $t = 1$ as well as the slope of the parametric curve between those two values of t.

a. $f(t) = (2 - 8t, 4t - 4)$

b. $f(t) = (t^2 + 1, 2t - 3)$

c. $f(t) = (|t - 2|, |t + 1|)$

d. $f(t) = (t^2 - 1, t^4 - 4t)$

7. Use your calculators to graph the curve represented by the parametric equations. Indicate the orientation of the curve. Identify any points at which the curve is not smooth

a. Cycloid: The curve traced by a point on the circumference of a circle as it rolls on a straight line.

$x = 2(t - \sin t)$ and $y = 2(1 - \cos t), t = 0, 4\pi$

Scale

[,]
[,]

b. Prolate Cycloid: Same as a) except the point goes below the line (railroad track)

$x = 2\theta - 4\sin\theta$ and $y = 2 - 4\cos\theta, t = 0, 6\pi$

Scale

[,]
[,]

c. Hypocycloid: $x = 3\cos^3 t$ and $y = 3\sin^3 t, t = 0, 2\pi$

Scale

[,]
[,]

d. Curtate cycloid: $x = 2t - \sin t$ and $y = 2 - \cos t, t = 0, 6\pi$

Scale

[,]
[,]

e. Witch of Agnesi: $x = 2\cot t$ and $y = 2\sin^2 t, t = 0, 2\pi$

Scale

[,]
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f. Folium: $x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}, t = 0, \infty$

Scale

[,]
[,]

6. A dart is thrown upward from 6 ft. high with an initial velocity of 18 feet/sec at an angle of elevation of $41°$.

a. Write a parametric equation that describes the position of the dart at time *t*.

b. Approximately how long will it take for the dart to hit the ground?

c. Find the approximate maximum height of the dart.

d. How long will it take for the dart to reach maximum height?

7. An arrow is shot from a platform 20 feet off the ground with an initial velocity of 150 feet/sec at an angle of elevation of $23°$.

a. Write a parametric equation that describes the position of the arrow at time *t*.

b. Find the approximate maximum height of the arrow.

c. Approximately how long will it take for the arrow to reach maximum height?

d. There is a wall 20 feet high 500 feet from the archer. Will the arrow hit it?

If so, how long will it take to hit it? If not, when will the arrow hit the ground beyond the wall and how far away will it land?

8. A golfer hits a ball with an initial velocity of 90 mph at angle of elevation of 64°.

a. Write a parametric equation that describes the position of the ball at time *t*.

b. Approximately how long will it take for the ball to hit the ground?

c. Find the approximate maximum height of the ball.

d. The green is 150 yards away. Will the ball reach the green? Explain.

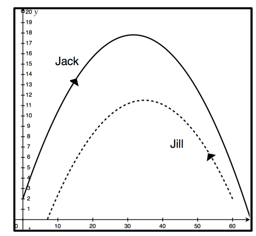
9. An NFL kicker at the 33-yard line attempts a field goal. The ball leaves his foot at 69 feet/sec at an angle of elevation of $38°$.

a. Write a parametric equation that describes the position of the ball at time *t*.

b. How high does the ball get above the field?

c. The goal posts are 10 feet high and are 43 yards away from him. If the kick is straight, is the field goal good? Explain.

10. Jack and Jill are standing 60 feet apart as shown in the figure to the right. At the same time, they each throw a softball from an initial height of two feet towards each other. Jack throws the softball at an initial velocity of 45 ft/sec at an angle of elevation of $44°$. Jill throws her ball with an initial velocity of 41 ft/sec with an angle of elevation of $37°$.



a. Write 2 parametric equations that describes the position of the ball at time *t*. Remember they are throwing the balls toward each other.

b. Find the heights of each ball.

c. About how far does each ball travel?

d. If neither catches the ball, when does each ball hit the ground?

e. Write a function that describes the distance between the two balls and use it to find this distance when the balls are closest together.

