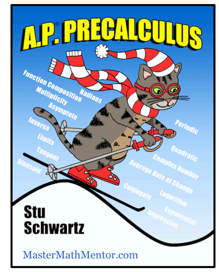
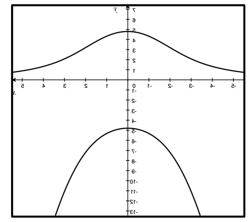
Topic 4.2 – The Conic Sections – Classwork



Implicitly Defined Functions

Consider the equation $x^2y + y^2 = 23$. We can see at a glance that the point (3, 2) satisfies it. So does the point $(-3, 2)$. Is it a function? Again, careful inspection shows that it is not. The point (3, –11) satisfies it as well so the relation fails the vertical line test.



To the right is the graph of the expression. How is this graph obtained? The software that is used for most of the graphs in this manual allows the graphing of equations expressed *implicitly*. An *implicit equation* expresses the relationship between *x* and *y* without solving for *y*.

When an equation is in implicit form, we can use it to find a *y*-value given an *x*-value. For instance, if *x* = 4, we get $y^2 + 16y = 23$ and by the quadratic equation we get $x = -8 \pm 87$.

Having the ability to graph equations implicitly is a great help. If we wanted to graph this on the TI-84, we would have to solve for *y,* placing the expression in *explicit* or *closed form*. This can be achieved by completing the square as shown to the right.

$$y^2 + x^2y = 23$$
$$y^2 + x^2y + \frac{x^4}{4} = 23 + \frac{x^4}{4}$$
$$\left(y + \frac{x^2}{2}\right)^2 = 23 + \frac{x^4}{4}$$
$$y + \frac{x^2}{2} = \pm\sqrt{23 + \frac{x^4}{4}}$$
$$y = \frac{-x^2 \pm \sqrt{92 + x^4}}{2}$$

So while it is possible, it is certainly not something we'd prefer to do unless we really needed to see a graph of the expression. And if the expression were changed to $x^2y + y^3 = 23$, it may be impossible to create a closed form expression. There are many such expressions: For instance, $y \sin y = x$, $y = e^{x+y}$, $y + \ln y = x + 2^x$ to name a few.

Suppose you do not have the ability to solve implicit equations or it is impossible. We can still use our calculator to find a point given an *x*-value. For instance, suppose $x^2y + y^3 = 23$ and we wanted the value of *y* when *x* = 4. Substituting, we get $y^3 + 16y = 23$. To solve this for *y*, simply solve the equation $x^3 + 16x = 23$ (there is only one variable so it can be written in terms of *x* rather than *y*) graphically by finding the zero(s) of $y = x^3 + 16x - 23$ and getting *x* =1.30. So the equation $x^2y + y^3 = 23$ passes through the point $(4, 1.30)$.

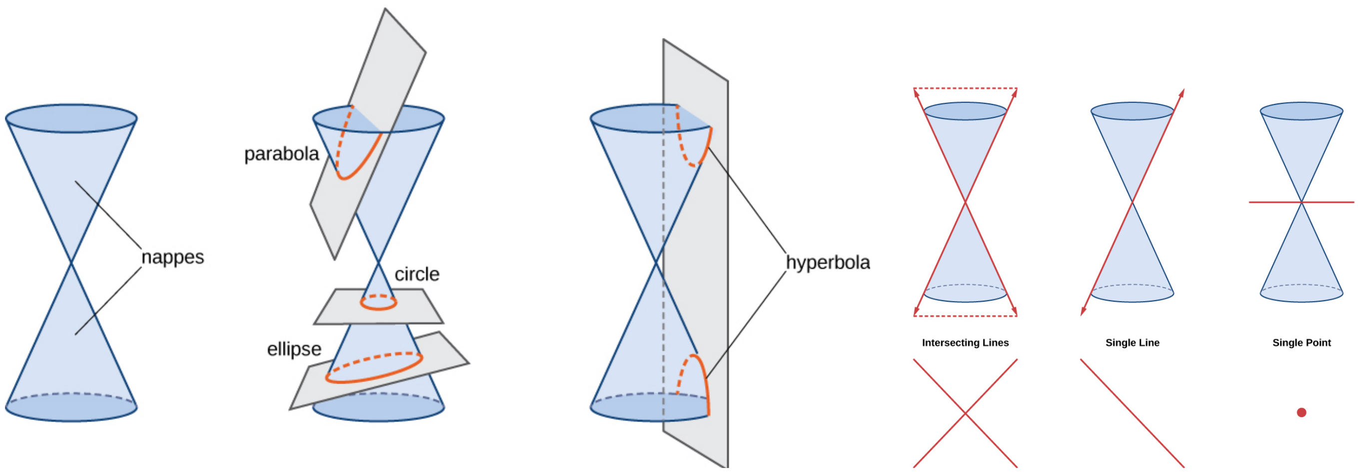
There are expressions we will leave in implicit form because they are easier to write, even though they could be solved explicitly More important, leaving them in implicit form gives use important information as to the nature of their graphs.

Example 1) For the following implicit curves, find the value(s) of *y* at the given value of *x* by finding an equation to solve graphically.

#	Equation	x – value	Equation to graph	y – value
a)	$x^2y^2 + xy = 20$	$x = 3$		
b)	$y + 2^y = 6x$	$x = 4$		
c)	$y + x\sin y = 2$	$x = -4$		
d)	$\left(x^2 + y^2\right)^2 = 2\left(x^3 + y^3\right)$	$x = 1$		

The Conic Sections

Conics are formed by the intersection of a plane and a double-napped cone. Depending on the slope of the plane, the resulting figure is one of the four basic conics. Each set of figures below shows the plane intersecting the cone with the resulting conic. When the plane goes through the vertex as it does in the last three figures, it is called a degenerate conic.



As we saw with lines, there are several algebraic forms of conics, the general form and the standard form. The general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where A, B, C, D, E, and F are integers. In this unit, we will examine conics in the form where $B = 0$. Unfortunately, as we saw with lines, general form tells you very little about the conic and thus, we will move onto standard form, while, not as "pretty", gives succinct information about the conic.

In this unit, problems will be of two types: a) Given information about the conic, find its equation and b) given the equation of the conic, find information about the conic. Completing the square is used extensively here.

The Circle

The simplest conic is the circle. Its definition is the set of all points equidistant from a point not on the circle (the center). This distance is called the radius.

Standard form of a circle is: $(x-h)^2 + (y-k)^2 = r^2$ where the center of the circle is (h,k) and the radius is *r*.
This is written in implicit form. Recall also that a circle can be written in parametric form:
$x = h \pm r\cos b \quad y = k \pm r\sin b$ where the center of the circle is (h,k) and the radius is *r*.

Example 2) Find the equation of the circle in general form given its center and radius.

a) $(0,0),\ r = 4$ b) $(-2,3),\ r = 8$ c) $\left(\frac{1}{2},\frac{-1}{4}\right),\ r = 1$

Example 3) Find the equation of the circle in general form with the following specifications.

a) Center $(-1,4)$, pt. on circle $(2,3)$ b) Diameter endpts. $(-1,4),(3,-8)$ c) Through $(7,6),(-1,2),(0,5)$

Example 4) Find the center and radius of the circles with the following equations

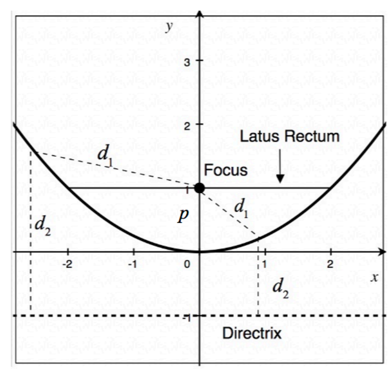
a) $x^2+y^2+10x-6y+4=0$ b) $x^2+y^2-7x+3y-8=0$ c) $4x^2+4y^2+16x-1=0$

The Parabola

A parabola is the set of all points (x,y) that are equidistant from a fixed line called the directrix and a fixed point called the focus, both not on the parabola.

In the parabola to the right, the distance d_1 from the focus to any point on the parabola is equal to the distance d_2 from the parabola to the directrix.

A simple way to sketch the parabola is to draw the line through the focus parallel to the directrix. This line is called the latus rectum and its length is 4*p* where *p* is the distance from the vertex to the focus.



If the vertex of the parabola lies at the origin and *p* is the directed distance from the vertex to the parabola, the standard equations of a parabola are:

$x^2=4py$ opens up ($p>0$) or down ($p<0$) and $y^2=4px$ opens right ($p>0$) or left ($p<0$).

If the parabola has a vertex at the point (h,k), the standard equations for the parabola are:

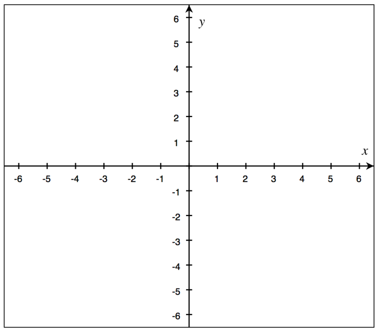
$(x-h)^2=4p(y-k)$ for a parabola opening up/down and $(y-k)^2=4p(x-h)$ for a parabola opening left/right.
Parabolas can be placed in parametric form solving an up/down equation for $y(t)$ and making $x=t$
For a left/right equation, solve for $x(t)$ and make $y=t$

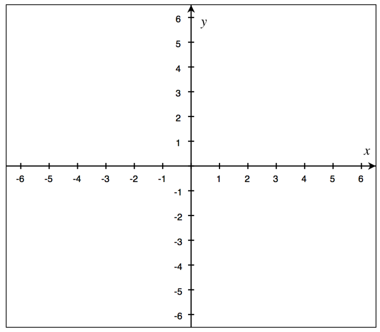
Example 5) Sketch and find the equation of the parabola with the given vertex and focus. Sketch and find the equation of the directrix as well.

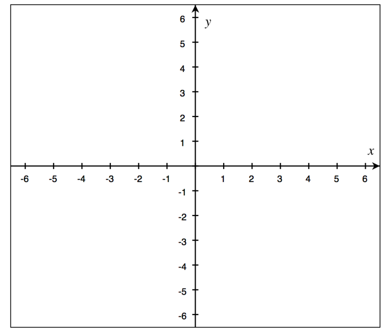
a) $V(0,0), F(0,2)$

b) $V(0,0), F(-2,0)$

c) $V(0,0), F\left(0,-\frac{1}{4}\right)$



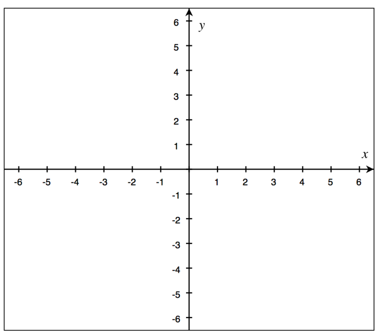


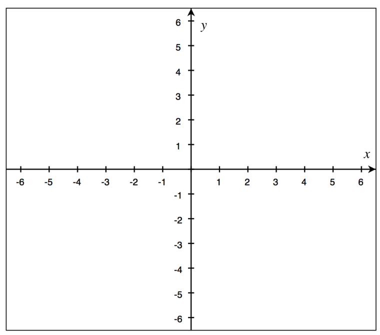


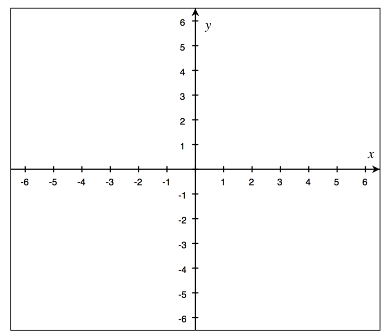
d) $V(2,3), F(2,2)$

e) $V(-1,2), F(1,2)$

f) $V(-3,-2), F(-3,-0.5)$





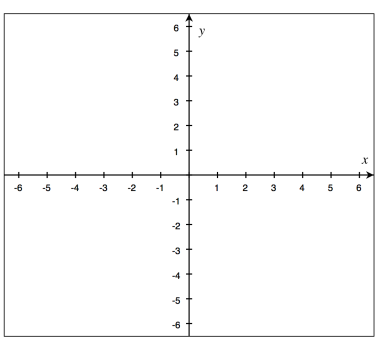


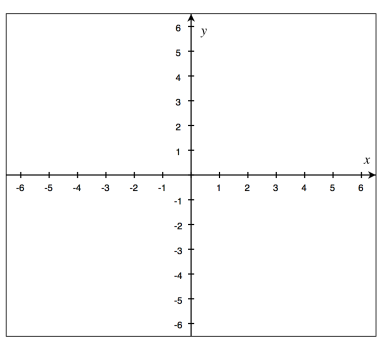
Example 6) Find the equation of the parabola with the given information.

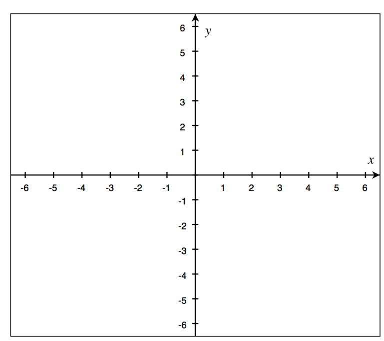
a) Vertex (–2, 3), directrix: $y = 6$

b) Focus (2, –1), Directrix: $x = -2$

c) Vertex (0,0), opens up
(2,16) on curve





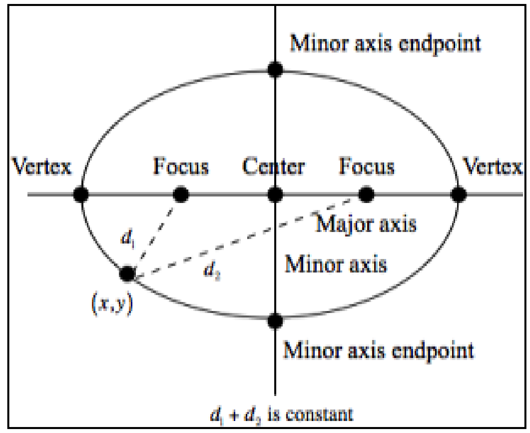


Example 7) Find the vertex, focus, and directrix of the parabola.

a) $x^2 + 12y = 0$ b) $x^2 - 4y - 4 = 0$ c) $y^2 - 2y + 8x + 9 = 0$

The Ellipse

An ellipse is the set of all points (x, y) the sum of whose distances from two fixed points called the foci is constant. The line through the foci intersects the ellipse at two points called vertices. The chord joining the vertices is the major axis and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is called the minor axis.



If an ellipse has center at the origin (0,0), its equation in standard form is:

Horizontal ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b$

The major axis lies on the x-axis.

Vertices are $(a,0),(-a,0)$

Major axis length $= 2a$

Minor axis endpoints: $(0,b),(0,-b)$

Minor axis length $= 2b$

Foci lie on the major axis at $(c,0),(-c,0)$

Vertical ellipse

$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ with $a > b$

The major axis lies on the y-axis.

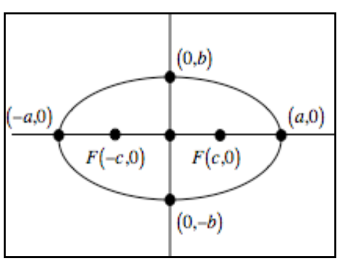
Vertices are $(0,a),(0,-a)$

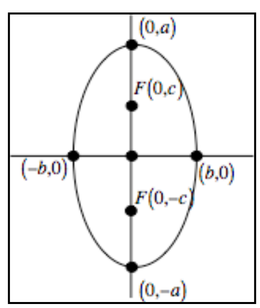
Major axis length $= 2a$

Minor axis endpoints: $(b,0),(-b,0)$

Minor axis length $= 2b$

Foci lie on the major axis at $(0,c),(0,-c)$





In both ellipses, the relationship between $a, b,$ and c is: $c^2 = a^2 - b^2$

The *eccentricity* of the ellipse is $e = c/a$. $0 \le e \le 1$. The closer to 0, the rounder the ellipse.

The closer to 1, the flatter the ellipse.

If the center of the ellipse is (h,k), the ellipse has equations:

Horizontal ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ Vertical ellipse: $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

The ellipse can be written parametrically: $x=h\pm a\cos t \quad y=k\pm b\sin t$

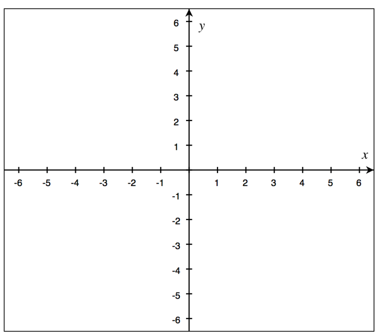
So a circle is just a special case of an ellipse with $a=b$

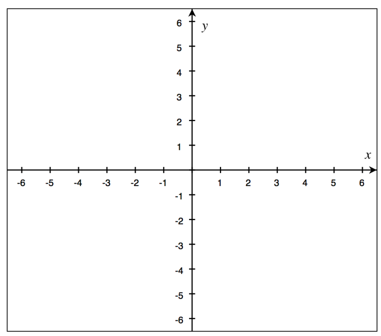
Example 8) Find the center, vertices, foci, and eccentricity of the ellipse and sketch it.

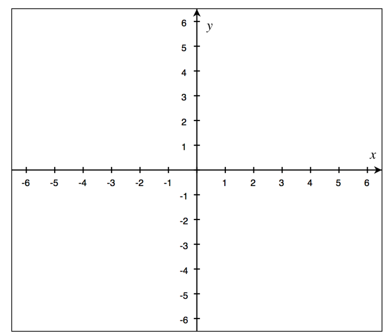
a) $\frac{x^2}{16}+\frac{y^2}{9}=1$

b) $\frac{x^2}{9}+\frac{y^2}{16}=1$

c) $\frac{x^2}{25}+\frac{y^2}{10}=1$



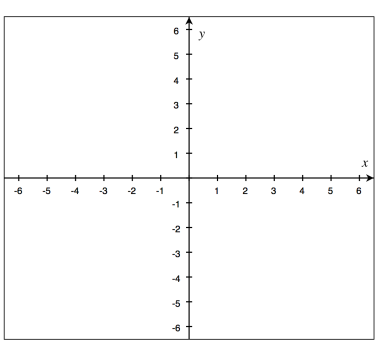


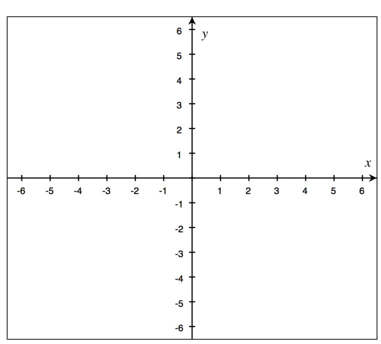


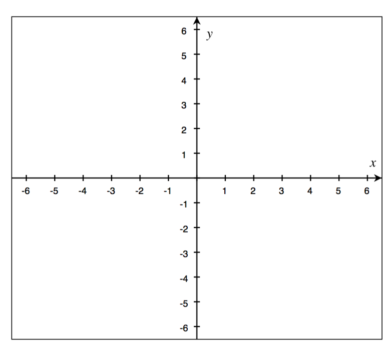
d) $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{4}=1$

e) $(x+3)^2+\frac{(y-1)^2}{4}=1$

f) $\frac{(x-1)^2}{9}+4(y+1)^2=1$



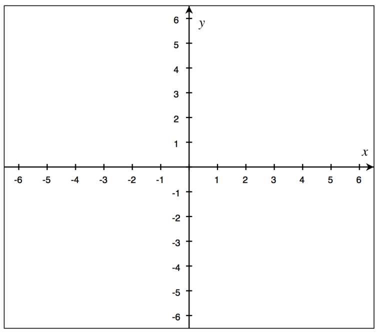


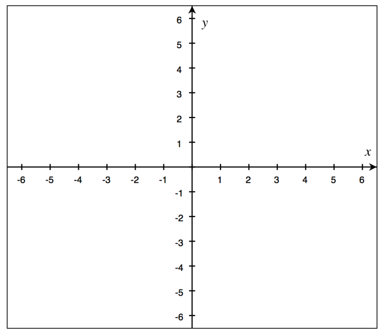


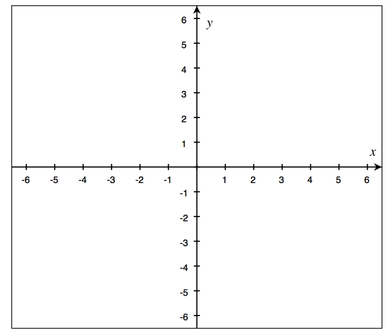
g) $x^2 - 2x + 4y^2 - 3 = 0$

h) $4x^2 + 16x + 9y^2 - 36y + 16 = 0$

i) $9x^2 + 72x + y^2 + 2y + 136 = 0$







Example 9) Find the equation of the ellipse with the given conditions.

a) Vertices $(\pm 2, 0)$, minor axis length = 2.

b) Vertices $(0, \pm 5)$, foci $(0, \pm 3)$

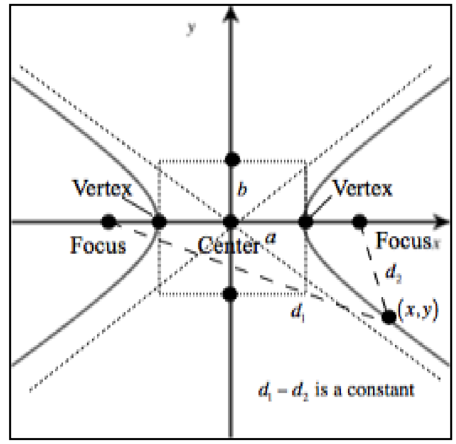
c) Center (2,3), Vertex (2,7) Focus (2,6)

d) Vertices (4,0), (4,16). Minor axis endpts. (–2,8), (10,8)

e) Vertices (2,3), (2,9) Eccentricity = $2/3$

f) Center (4, –3), Vertex (1, –3) Focus $\left(4 - \sqrt{2}, -3\right)$

The Hyperbola



The hyperbola parallels the ellipse in meaning. In the ellipse, the sum of distances from two points called the foci are a constant. The hyperbola is the set of all points (x,y) the difference of whose distances from two fixed points called the foci is constant. Every hyperbola has two disconnected branches. The line through the two foci intersects a hyperbola at its two vertices. The line segment connecting the two vertices is called the transverse axis. The midpoint of the transverse axis is called the center of the hyperbola. The axis perpendicular to the transverse axis through the center is called the conjugate axis. If an hyperbola has center at the origin (0,0), its equation in standard form is:

Horizontal Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ a and b have no size relationship

The transverse axis lies on the x-axis.

Vertices are $(a,0),(-a,0)$

Transverse axis length $= 2a$

Conjugate axis endpoints: $(0,b),(0,-b)$

Conjugate axis length $= 2b$

Foci lie on the transverse axis at $(c,0),(-c,0)$

Vertical Hyperbola

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ a and b have no size relationship

The transverse axis lies on the y-axis.

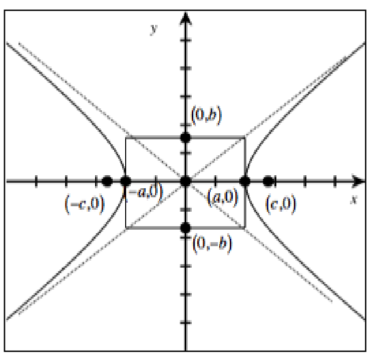
Vertices are $(0,a),(0,-a)$

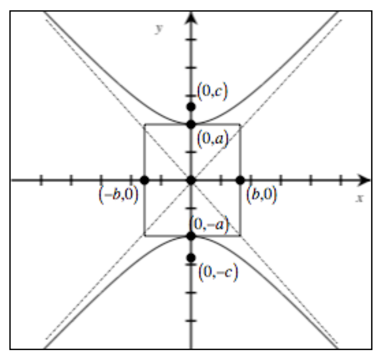
Transverse axis length $= 2a$

Conjugate axis endpoints: $(b,0),(-b,0)$

Conjugate axis length $= 2b$

Foci lie on the transverse axis at $(0,c),(0,-c)$





In both hyperbolas, the relationship between a, b, and c is $c^2 = a^2 + b^2$ (different than the ellipse). The eccentricity of the hyperbola is $e = c/a$. (same as the ellipse). But since $c > a$, $e > 1$. The closer to 1, the flatter the hyperbola. The larger the value of e, the rounder the hyperbola is at its vertices.

Hyperbolas are easily sketched by first determining whether the ellipse is horizontal or vertical. If the x-fraction is positive, it is horizontal. If the y-fraction is positive, it is vertical. Note that, unlike the ellipse, the relative values of a and b have no determination on whether the hyperbola is horizontal or vertical.

Once you have the orientation of the hyperbola, place points at the vertices and at the conjugate axis endpoints. Draw a rectangle connecting these points with sides parallel to the transverse and conjugate axis. Lightly draw two dotted-lines through the diagonals. These are asymptotes of the hyperbola, and like asymptotes we studied before, will be lines that the hyperbola will approach far from the center.

The equations for the asymptotes (center at the origin) is $y = \frac{b}{a}x$ for horizontals and $y = \frac{a}{b}x$ for verticals.

Finally, if the hyperbola has center (h,k), the standard equations of the hyperbola are:

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for horizontal hyperbolas and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ for vertical hyperbolas.

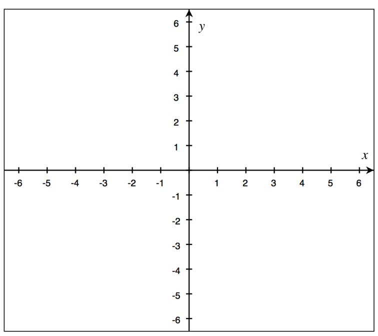
An hyperbola can be written in parametric form: $\begin{cases} \text{opening left-right: } x = h + a\sec t \quad y = k + b\tan t \\ \text{opening up-down: } x = h + a\tan t \quad y = k + b\sec t \end{cases}$

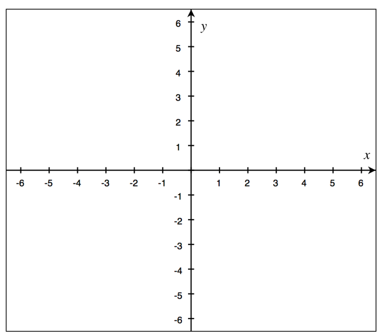
Example 10) Find the center, vertices, foci, and eccentricity of the hyperbola and sketch it with its asymptotes.

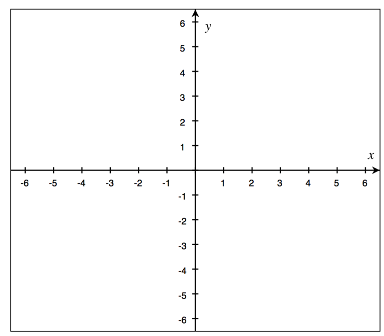
a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b) $\frac{y^2}{9} - \frac{x^2}{16} = 1$

c) $x^2 - \frac{y^2}{9} = 1$



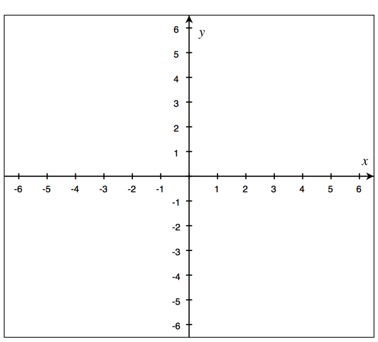


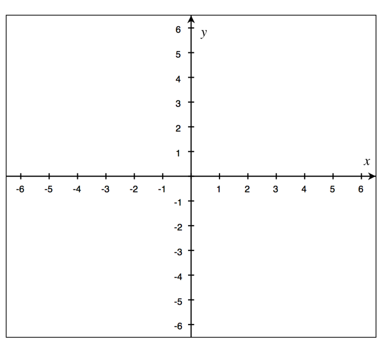


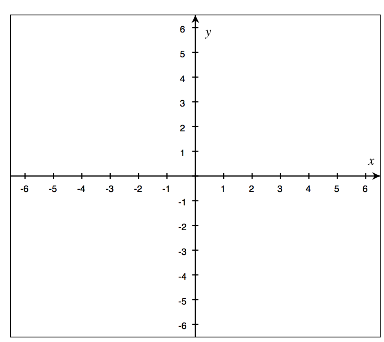
d) $\frac{(x-2)^2}{4} - \frac{(y+1)^2}{9} = 1$

e) $\frac{(y-1)^2}{4} - \frac{(x+3)^2}{4} = 1$

f) $\frac{(x-1)^2}{9} - 4(y+1)^2 = 1$



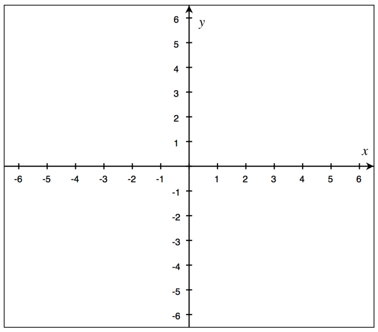


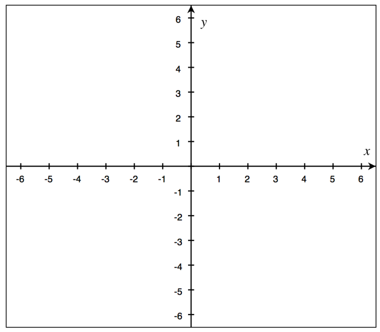


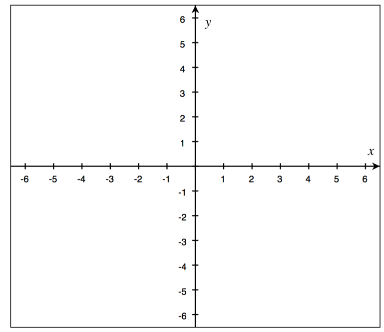
g) $x^2 - 2x - 4y^2 - 3 = 0$

h) $4x^2 + 16x - 9y^2 + 36y - 56 = 0$

i) $4y^2 + 8y - x^2 - 6x - 21 = 0$







Example 11) Find the equation of the hyperbola with the given conditions.

a) Vertices $(\pm 2, 0)$, Foci $(\pm 3, 0)$

b) Vertices $(0, \pm 5)$, foci $(0, \pm 8)$

c) Vertices $(\pm 1, 0)$,
Asymptotes $y = \pm 4x$

d) Vertices (0,2), (0,6)
Foci (0,–1), (0, 9)

e) Vertices (5,3), (5,–5)
Eccentricity = $3/2$

f) Vertices (1, –2), (1,2)
Passes through (3, 4)

Determining Conic Type from General Equation

Earlier, we said that any equation in the form of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ generates a conic. While true, some conics are what are called degenerate, meaning that they do not form a circle, parabola, ellipse, or hyperbola.

Examples: If $A = 1, C = 1, D = 0, E = 0, F = 0$, the equation $x^2 + y^2 = 0$ simply graphs the point (0, 0).
If $A = 0, C = 0, D = 1, E = 1, F = -2$, the equation $x + y - 2 = 0$ graphs a line, $y = 2 - x$.
If $A = 4, C = -1, D = 0, E = 0, F = 0$, the equation $4x^2 - y^2 = 0$ gives two lines, $y = \pm 2x$.
If $A = 1, C = 1, D = 0, E = 0, F = 1$, the equation $x^2 + y^2 + 1 = 0$ graphs no points at all.

Assuming that we have a conic that is not degenerate, there is an easy test to determine the actual shape of the conic without going through the trouble of completing the squares to put it into standard form.

If $Ax^2 + Cy^2 + Dx + Ey + F = 0$ generates a conic, it is one of the following

- If $A = C$, it generates a circle
- If $AC > 0$ (meaning A and C have like signs), it generates an ellipse
- If $AC < 0$ (meaning A and C have unlike signs), it generates an hyperbola
- If $AC = 0$ ($A = 0$ or $C = 0$ but not both), it is a parabola

Example 12) Classify each graph and for those that are not degenerate, write in parametric form.

a) $2x^2 + 3y^2 + 4x - 6y + 1 = 0$

b) $9x^2 - 9y^2 - 3x - 2y + 3 = 0$

c) $x^2 - 4x - 2y + 3 = 0$

d) $y^2 - 2x^2 - 8y - 12z - 18 = 0$

e) $2x^2 + 6x + 2y^2 - 10y - 3 = 0$

f) $x^2 + 10x + y^2 - 8y + 100 = 0$

Topic 4.2 – The Conic Sections – Homework

1. For the following implicit curves, find the value(s) of y at the given value of x by finding an equation to solve graphically.

#	Equation	$x -$ value	Equation to graph	$y -$ value
a.	$xy^2 - x^2y = 6$	$x = 2$		
b.	$\sqrt{xy} = x - 2y$	$x = 4$		
c.	$2y^3 + 4x^2 - y = 3x^3$	$x = 1$		
d.	$y^2e^{2x} = 3y + x^2$	$x = 3$		

2. Find the equation of the circle in general form given its center and radius.

a. $(0,1)$, $r = 6$

b. $(5,-4)$, $r = 10$

c. $\left(\frac{-3}{4},\frac{3}{2}\right)$, $r = \frac{1}{2}$

3. Find the equation of the circle in general form with the following specifications.

a. Center $(3,-9)$, pt. on circle $(4,-1)$

b. Diameter endpts. $(8,-6),(-2,10)$

c. Through $(-3,-2),(1,-7),(-4,-3)$

4. Find the center and radius of the circles with the following equations.

a. $x^2 + y^2 - 4x + 18y + 49 = 0$

b. $x^2 + y^2 + 9y + 1 = 0$

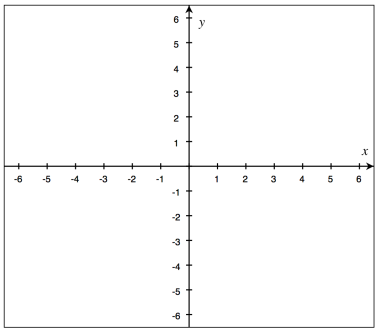
c. $2x^2 + 2y^2 + 4x - y = 0$

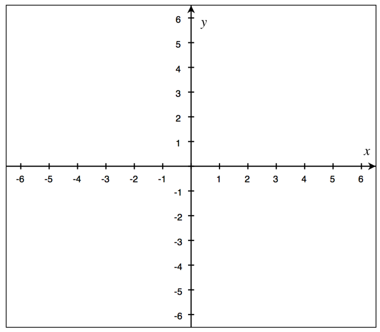
5. Sketch and find the equation of the parabola with the given vertex and focus. Sketch and find the equation of the directrix as well.

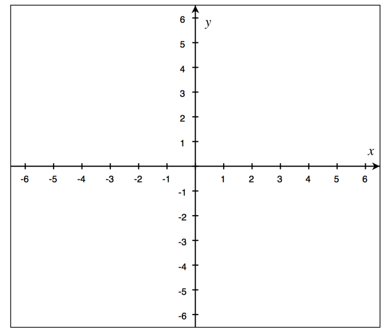
a. $V(0,0), F(-3,0)$

b. $V(0,0), F(0,1.5)$

c. $V(0,0), F\left(\frac{1}{8},0\right)$



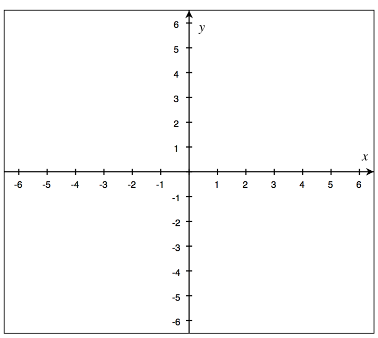


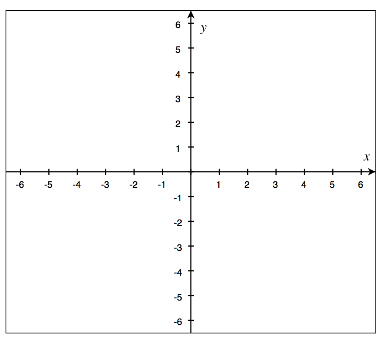


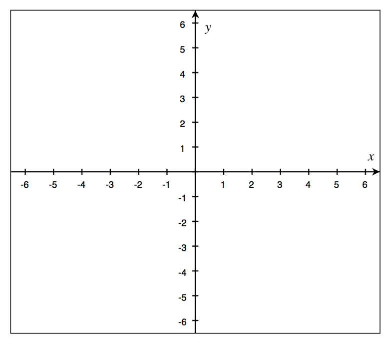
d. $V(4,1), F(4,2)$

e. $V(2,-3), F(0,-3)$

f. $V(1,5), F(1,4.5)$





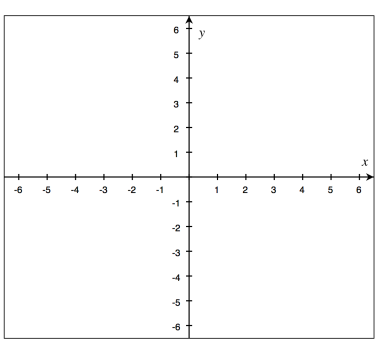


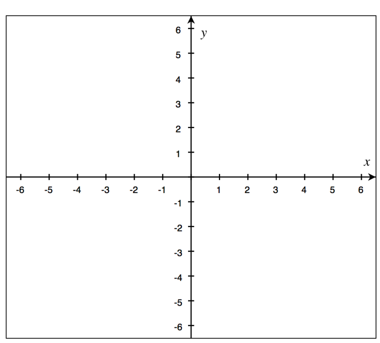
6. Find the equation of the parabola with the given information.

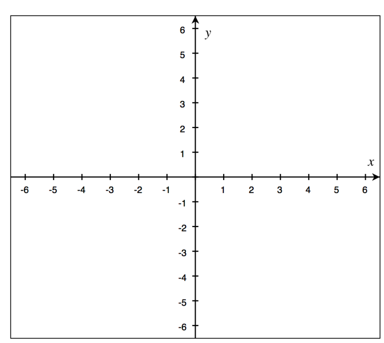
a. Vertex (–1, –3), directrix: $y = -6$

b. Focus (1, 1), Directrix: $x = 5$

c. Vertex (1,4), opens down
(0,–4) on curve







7. Find the vertex, focus, and directrix of the parabola.

a. $y^2 - 20x = 0$

b. $x^2 + 6y - 24 = 0$

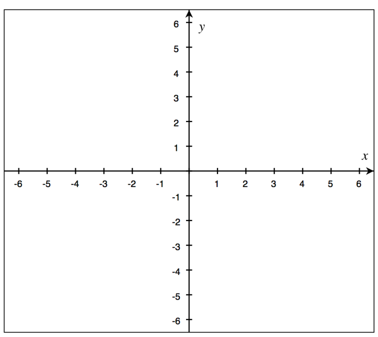
c. $x^2 + 2x - 12y - 59 = 0$

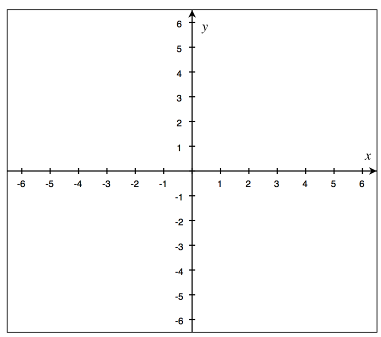
8. Find the center, vertices, foci, and eccentricity of the ellipse and sketch it.

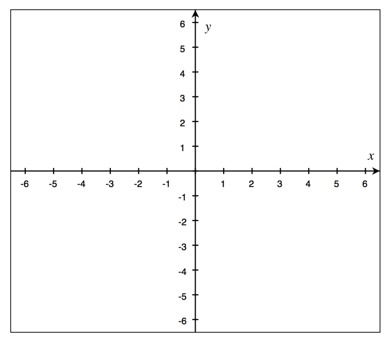
a. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

b. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

c. $\dfrac{x^2}{4} + \dfrac{y^2}{2} = 1$



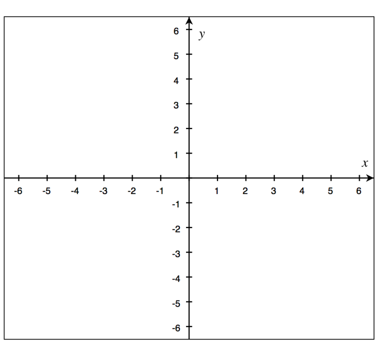


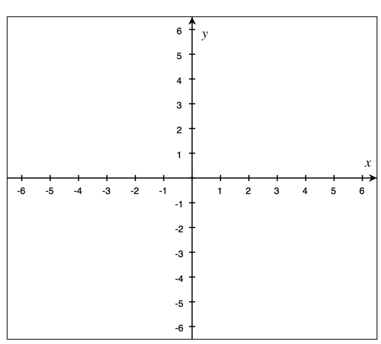


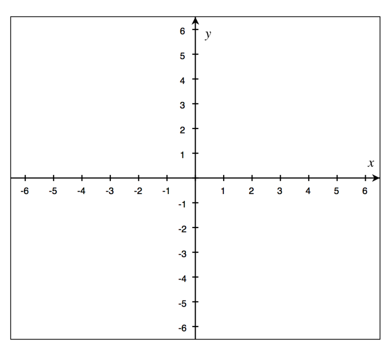
d. $\dfrac{(x+1)^2}{4} + \dfrac{(y-2)^2}{16} = 1$

e. $\dfrac{(x-2)^2}{9} + (y+1)^2 = 1$

f. $4x^2 + \dfrac{(y-1)^2}{16} = 1$



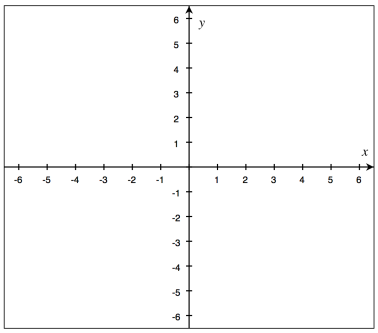


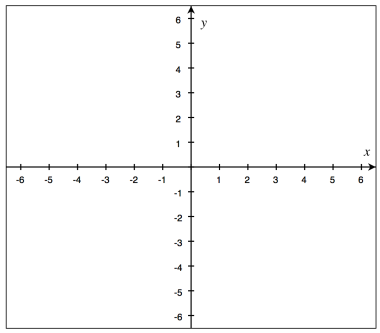


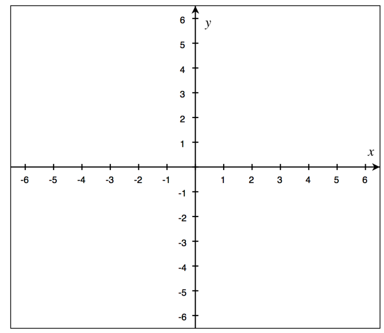
g. $4x^2 + 25y^2 - 100y = 0$

h. $4x^2 + y^2 - 32x - 6y + 69 = 0$

i. $x^2 + 4x + 64y^2 - 12 = 0$







9. Find the equation of the ellipse with the given conditions.

a. Vertices $(\pm 9,0)$, minor axis length = 7

b. Vertices $(0,\pm 5)$, foci $(0,\pm 4)$

c. Center (–5,4), Vertex (1,4) Focus (0,4)

d. Vertices (0,–9), (0,13). Minor axis endpt. (5,2)

e. Vertices (–10,5), (2,5) Eccentricity = $1/6$

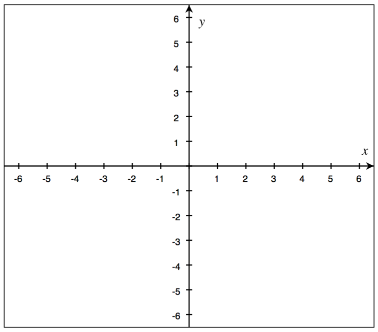
f. Center (–3,–7), Vertex (–3,3) Focus $\left(-3,-7+\sqrt{5}\right)$

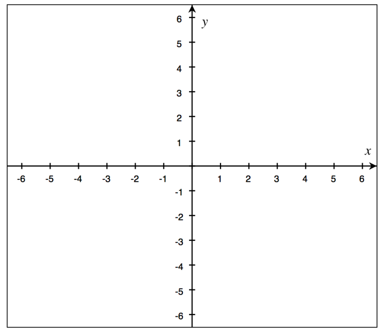
10. Find the center, vertices, foci, and eccentricity of the hyperbola and sketch it with its asymptotes.

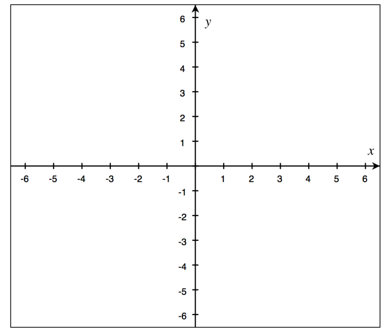
a. $y^2 - \dfrac{x^2}{4} = 1$

b. $x^2 - \dfrac{y^2}{4} = 1$

c. $\dfrac{x^2}{9} - y^2 = 1$



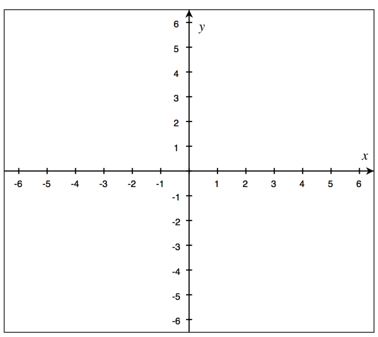


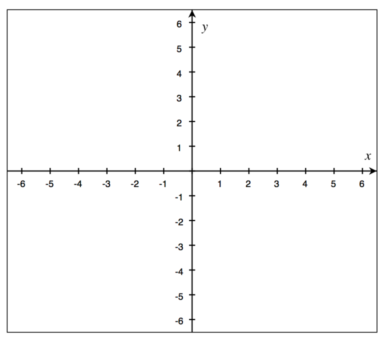


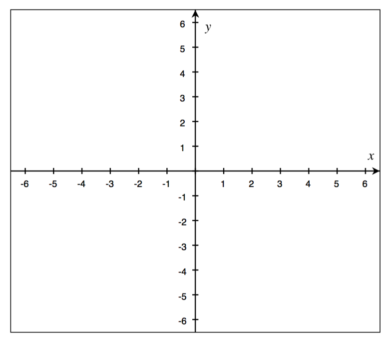
d. $\dfrac{(x+1)^2}{4} - \dfrac{(y-2)^2}{9} = 1$

e. $\dfrac{(y+1)^2}{9} - \dfrac{x^2}{9} = 1$

f. $(x+2)^2 - 9(y-2)^2 = 1$



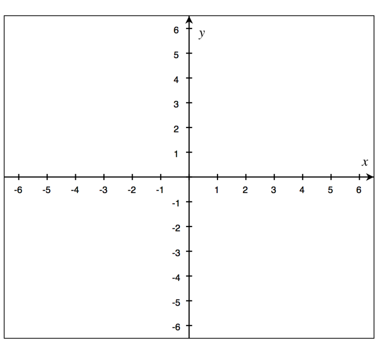


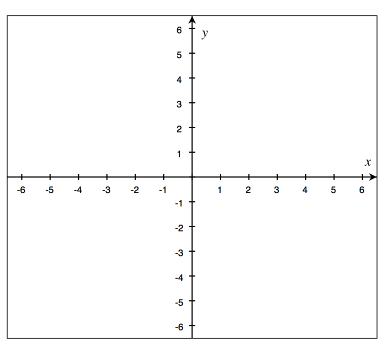


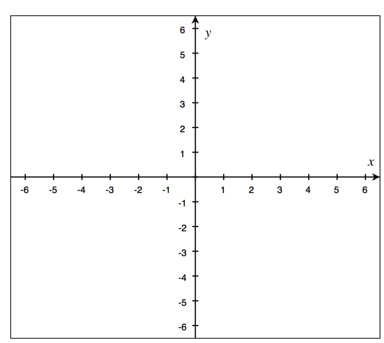
g. $9x^2 - y^2 - 4y - 13 = 0$

h. $y^2 - 4y - x^2 - 6x - 9 = 0$

i. $x^2 - 2x - 36y^2 - 72y - 39 = 0$







11. Find the equation of the hyperbola with the given conditions.

a. Vertices $(\pm 5,0)$, Foci $(\pm 6,0)$

b. Vertices $(0,\pm 1)$, Foci $(0,\pm 9)$

c. Vertices $(\pm 3,0)$, Asymptotes $y=\pm\frac{2x}{3}$

d. Vertices (1,–2), (1,6) Foci (1,–4), (1, 8)

e. Vertices (–6,–7), (–6,–5) Eccentricity = 2

f. Vertices (3,2), (–3,2) Passes through (5,0)

12. Classify each graph and for those that are not degenerate, write in parametric form.

a. $5y^2+20y-2x-6=0$

b. $4x^2-9y^2+40x+18y-53=0$

c. $2y^2-2x^2+4y+4x=0$

d. $6x^2+6y^2-54x+90y-455=0$

e. $2x^2+6y^2+6x-6y-3=0$

f. $x^2+10x+y^2-8y+100=0$