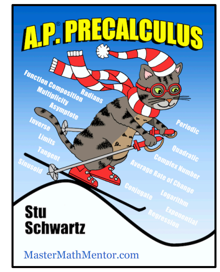
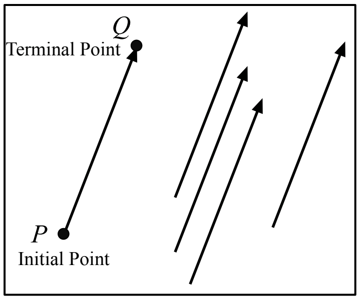
Topic 4.3 – Vectors – Classwork





In geometry and physics, concepts such as temperature, mass, length, area, and volume can be quantified with a single real number. These are called *scalar quantities* and the real number associated with it is called a *scalar*. But force, velocity, and acceleration involve both magnitude (size) and direction and cannot be characterized by a single number. To represent these quantities, we use a directed line segment as shown to the right. The directed line segment PQ has initial point P and terminal point Q and we denote its length or *magnitude* by $\|PQ\|$. Two directed line segments that have the same magnitude and direction are called equivalent. Vectors that have the same direction are parallel. All the vectors to the left are equivalent. We call each a *vector in the plane* and write $\mathbf{v} = PQ$. Vectors are usually denoted by the lower-case boldfaced letters, **u**, **v**, and **w**.

Example 1) Let **u** be the directed line segment from (0, 0) to (5, 2) and let **v** be the directed line segment from (–1, 3) to (4, 5). Show that **u** = **v**.

A line segment whose initial point is the origin and whose terminal point is (v_1, v_2) is given by the component form of **v** written as $\mathbf{v} = \langle v_1, v_2 \rangle$. The numbers v_1 and v_2 are called the components of **v**. To convert directed line segments to component form, use the following:

- If $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then **v** represented by PQ is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$. The magnitude (length) of PQ is $\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$. This means that we don't need an initial point and terminal point to represent a vector.
- If $\mathbf{v} = \langle v_1, v_2 \rangle$, then **v** can be represented by the directed line segment in standard position from P (0, 0) to Q (v_1, v_2). This means that we can move all vectors to the origin as initial point.

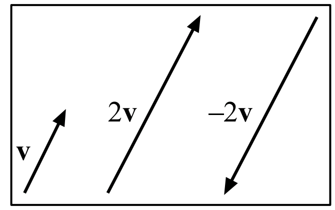
Example 2) Given an initial point and terminal point, write each as a vector in component form and find its magnitude.

a) Initial (3, 8), Terminal (6, 12)

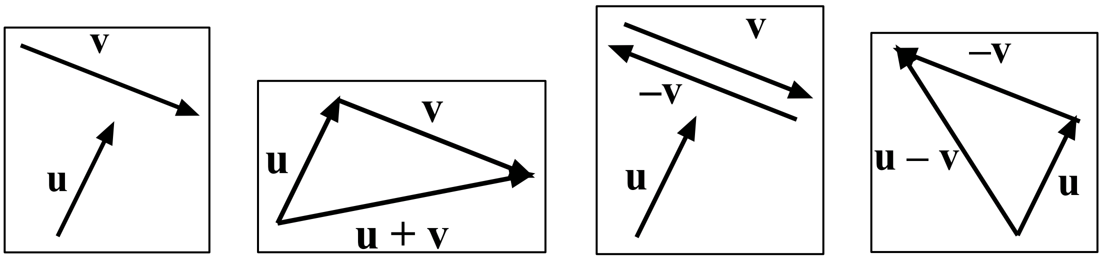
b) Initial (4, –6), Terminal (–1, 2)

c) Initial $\left(\frac{1}{2}, \frac{1}{3}\right)$ Terminal $Q\left(\frac{-1}{2}, \frac{-1}{6}\right)$

Geometrically, the product of a vector **v** and a scalar k is the vector that is k times as long as **v**. If k is positive, then the vector $k\mathbf{v}$ has the same direction as **v**. If k is negative, then the vector $k\mathbf{v}$ has the opposite direction as **v**.



To add two vectors, we draw them head to tails. The sum **u** + **v**, called the resultant vector is formed by joining the initial point of the first vector to the terminal point of the second vector as shown in the first two figures below. To subtract two vectors, we add the negative of the second vector as shown in the 3rd and 4th figures below.



For vectors $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ and scalar k, the following operations are defined:

- The scalar multiple of k and vector **u** is the vector $k\mathbf{u}=k\langle u_1,u_2\rangle=\langle ku_1,ku_2\rangle$.
- The vector sum of **u** and **v** is the vector $\mathbf{u}+\mathbf{v}=\langle u_1+v_1,u_2+v_2\rangle$.
- The negative of vector **v** is the vector $-\mathbf{v}=\langle -v_1,-v_2\rangle$.
- The vector difference of **u** and **v** is the vector $\mathbf{u}-\mathbf{v}=\langle u_1-v_1,u_2-v_2\rangle$.
- The magnitude of vector $k\mathbf{v}$ is $\|k\mathbf{v}\|=|k|\,\|\mathbf{v}\|$.

Example 3) Given the vectors $\mathbf{u}=\langle -3,7\rangle$ and $\mathbf{v}=\langle 5,1\rangle$, find the following:

a) $\frac{-1}{2}\mathbf{u}$ b) $\left\|\frac{-1}{2}\mathbf{u}\right\|$ c) $\mathbf{u}+\mathbf{v}$ d) $\|\mathbf{u}+\mathbf{v}\|$

e) $\mathbf{u}-\mathbf{v}$ f) $\|\mathbf{u}-\mathbf{v}\|$ g) $3\mathbf{u}-4\mathbf{v}$ h) $\|3\mathbf{u}-4\mathbf{v}\|$

A <u>unit vector</u> is a vector in the same direction as the original vector but it has magnitude 1. If **v** is a nonzero vector, then the vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

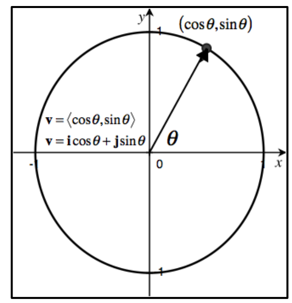
Example 4) Find unit vectors for the following and show that they have magnitude 1.

a) **u** b) **v** c) $\mathbf{u}+\mathbf{v}$

The unit vectors $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$ are called the *standard unit vectors* and are denoted by $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$. These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle = v_1 \langle 1,0 \rangle + v_2 \langle 0,1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$. We call $v_1\mathbf{i} + v_2\mathbf{j}$ a *linear combination* of **i** and **j**. The scalars v_1 and v_2 are called the components of **v**.

Example 5) Let **u** be the vector with initial points (–3, 7) and terminal point (–5, 2). Let **v** = 3**i** – 2**j**. Write the following as a linear combination of **i** and **j**.

a) **u** b) $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$ c) the unit vector for **w**



If **v** is a unit vector such that θ is the angle from the origin to the point $(\cos\theta, \sin\theta)$ lying on the unit circle, then $\mathbf{v} = \langle \cos\theta, \sin\theta \rangle = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$.

If **v** is any other vector such that θ is the angle from the origin to the point $\|\mathbf{v}\|(\cos\theta, \sin\theta)$, then $\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\|\mathbf{i}\cos\theta + \|\mathbf{v}\|\mathbf{j}\sin\theta$.

Example 6) Put the vectors with given magnitude and angle with *x*-axis in component form:

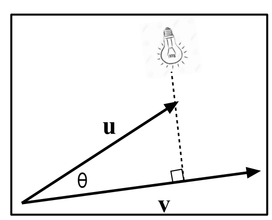
a) magnitude 6, angle of $60°$ b) magnitude 8, angle of $135°$

Dot product of two vectors

Multiplying two vectors is different than adding or subtracting vectors. When we add or subtract vectors, we get another vector. But when we multiply two vectors, we get a scalar.

The *dot product* of two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$. Note the result is a scalar. Simply multiply the two components and add.

The geometric interpretation of the dot product is the length of the projection of **u** onto **v**. If there were a light shining from a direction perpendicular to **v**, the dot product is the length of the shadow of **u** onto **v**. If the dot product is zero, there is no projection of **u** onto **v**. If the dot product is negative, the projection of **u** onto **v** is on the opposite direction of **v**.



Example 7) Given $\mathbf{u} = \langle 2, -2 \rangle$, $\mathbf{v} = \langle 5, 8 \rangle$, find

a) $\mathbf{u} \cdot \mathbf{v}$ b) $\mathbf{u} \cdot (2\mathbf{v})$ c) $\mathbf{u}^2$ d) $\mathbf{u}\mathbf{v}^2$

Definition of *orthogonal vectors*: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are called orthogonal (perpendicular) if $\mathbf{u}\cdot\mathbf{v} = 0$. Perpendicular, normal, and orthogonal all mean the same thing. However, we usually say that vectors are orthogonal, lines are perpendicular, and lines and curves are normal.

Angle between two vectors: If θ is the angle between two non-zero vectors **u** and **v**, then $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|}$

Note that by cross-multiplying you get that $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cos\theta$ which is another of finding the dot-product of two vectors, further emphasizing that the dot-product of two vectors is a scalar.

Example 8) Given $\mathbf{u} = \langle 3,-1\rangle$, $\mathbf{v} = \langle 4,3\rangle$, $\mathbf{w} = \left\langle \dfrac{2}{3},-\dfrac{8}{9}\right\rangle$ determine the angle between the two vectors and if any of the vectors are orthogonal.

a) **u** and **v**

b) **u** and **w**

c) **v** and **w**

Modeling Vectors

Vector word problems usually are in the form of two (or more) forces applied to an object. With the question being the magnitude and direction of the resultant force. The general methodology to solve these problems is as follows:

- Write force one as: $F_1\cos\theta_1\mathbf{i} + F_1\sin\theta_1\mathbf{j}$
- Write force two as $F_2\cos\theta_2\mathbf{i} + F_2\sin\theta_2\mathbf{j}$
- Add up each component (the **i** component and the **j** component) separately. You will get numbers for each so your answer will be in the form $a\mathbf{i} + b\mathbf{j}$
- The magnitude of the resultant vector will be the magnitude of $a\mathbf{i} + b\mathbf{j} = \sqrt{a^2 + b^2}$
- The direction of the resultant vector will be $\tan^{-1}\left(\dfrac{b}{a}\right)$.
- It is recommended that you make a sketch in order to visualize all the forces.

Example 9) I push a heavy desk by applying a force of 150 pounds at $25°$ to the desk. How much force is actually used in pushing the desk? What would happen if I increased the angle to $35°$?

Example 10) Two tugboats are pushing an ocean liner at angles of 17º to the liner to the northeast and southeast. What is the resultant force on the ocean liner if both boats pull with a force of 500 tons.

Example 11) Two people are pushing a piano. One person pushes it with 175 pounds at an angle of $60°$ to the x–axis while the other pushes it with 250 pounds at an angle of $30°$ to the x-axis. In what direction does the piano move and with how much force?

Example 12) I am in a motorboat crossing a river with a current of 4 mph. The motorboat travels perpendicular to the current at 12 mph. What is the resultant speed of the boat and at what angle do I hit the opposite riverbank?

Example 13) A plane traveling 500 mph (called airspeed) in the direction 150º encounters a wind of 80 mph in the direction of 45º. What is the resultant speed (called groundspeed) and direction of the plane?

Vector-Valued Functions

As an object moves along a curve in the plane, its coordinates x and y are both functions of time t. Rather than use f and g to represent these two functions, we write them as $x(t)$ and $y(t)$. These of course are parametric equations. To simplify further, we now describe the position as a *vector-valued function* in the form of $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

The magnitude of this position vector at time t gives the distance of the particle from the origin. We can also describe the velocity of the particle as a vector: $\mathbf{v}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Describe the motion of the object by examining $\mathbf{v}(t)$ to the right.

$x(t)$	$-$moving left	0 stopped	$+$ moving right
$y(t)$	$-$moving down	0 stopped	$+$ moving up

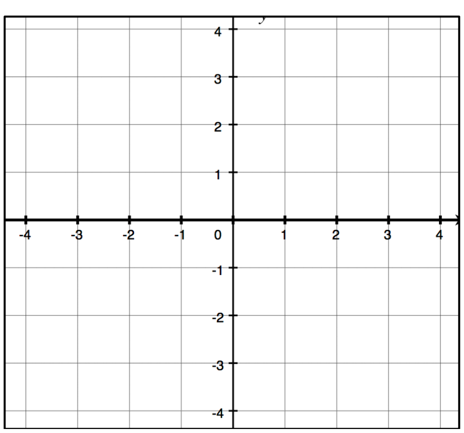
For an object to be fully stopped, both $x(t)$ and $y(t)$ must equal zero

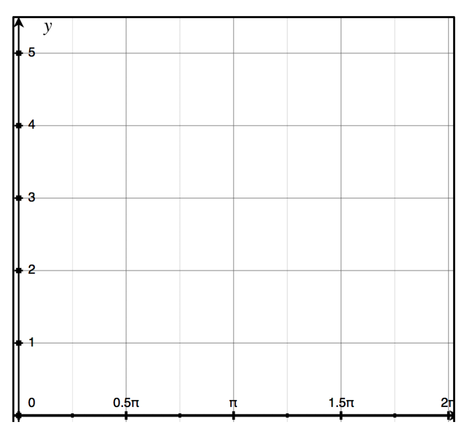
The speed at any time t = $\sqrt{[x(t)]^2 + [y(t)]^2}$

Example 14) A particle moves along a plane curve described by $\mathbf{r}(t) = 3\sin\left(\frac{t}{2}\right)\mathbf{i} + 3\cos\left(\frac{t}{2}\right)\mathbf{j}$.

a) Describe the path of the particle. b) The velocity vector is given by $\mathbf{v}(t) = \frac{3}{2}\cos\frac{t}{2}\mathbf{i} - \frac{3}{2}\sin\frac{t}{2}\mathbf{j}$. Find the speed of the particle at any time t.

Example 15) An object moves with position given by $\mathbf{r}(t) = (4\cos t)\mathbf{i} - (2\sin t)\mathbf{j},\ 0 \le t \le 2\pi$. a) Sketch the path of the object. b) The velocity of the object is $\mathbf{v}(t) = (-4\sin t)\mathbf{i} + (2\cos t)\mathbf{j}$. Draw the velocity vectors on your graph at $t = 0, \pi/2, \pi$ and $3\pi/2$. c) sketch a graph of the speed function as a function of t.





Topic 4.3 – Vectors – Homework

1. In the following, sketch the vector, then move it and write it in component form.

a. Initial pt. (1, 3)
terminal pt. (5, 5)

b. Initial pt. (4, –3)
terminal pt. (–2, –4)

c. Initial pt. $(-2.5, -0.5)$
terminal pt. $\left(-4, 4\tfrac{2}{3}\right)$

2. Determine if the vector **v** with initial point (p_1, p_2) and terminal point (q_1, q_2) is equivalent to vector **w** with initial point (r_1, r_2) and terminal point (s_1, s_2)

a. $\mathbf{v}\ (4,2),(-3,1)$
$\mathbf{w}(5,-3),(-2,-4)$

b. $\mathbf{v}\ (-1,6),(-4,4)$
$\mathbf{w}(2,-7),(4,-4)$

c. $\mathbf{v}\ (-7,0),(2,-9)$
$\mathbf{w}(5,-3),(-4,6)$

3. In the following, find the vector **w** where $\mathbf{u} = \langle 5,-3 \rangle$ and $\mathbf{v} = \langle 1,4 \rangle$

a. $\mathbf{w} = \dfrac{3}{4}\mathbf{u}$

b. $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$

c. $\mathbf{w} = \dfrac{-\mathbf{v} - \mathbf{u}}{3}$

d. $\mathbf{w} = \dfrac{1}{2}(-5\mathbf{u} + 3\mathbf{v})$

4. In the following, find *a* and *b* such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ where $\mathbf{u} = \langle 1,2 \rangle$ and $\mathbf{v} = \langle 1,-1 \rangle$

a. $\mathbf{w} = \langle 2,1 \rangle$

b. $\mathbf{w} = \langle 0,3 \rangle$

c. $\mathbf{w} = \langle -1,7 \rangle$

5. In the following, given that $\mathbf{u} = \langle 2,-3 \rangle$ and $\mathbf{v} = \langle 4,4 \rangle$, find the following:

a. $\|\mathbf{u}\|$

b. $\|\mathbf{v}\|$

c. $\|\mathbf{u} + \mathbf{v}\|$

d. $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$

e. $\left\| \dfrac{\mathbf{u}-\mathbf{v}}{\|\mathbf{u}-\mathbf{v}\|} \right\|$ f. $\mathbf{u}\cdot\mathbf{v}$ g. $\mathbf{u}\mathbf{v}^2$ h. $\dfrac{\mathbf{u}+\mathbf{v}}{\mathbf{u}^2\mathbf{v}^2}$

6. Find a unit vector in the direction of the following vectors and show that it has length 1.

a. $\mathbf{v}=\langle 5,12\rangle$ b. $\mathbf{v}=\langle 0,-2\rangle$ c. $\mathbf{v}=\langle -3,-3\rangle$

d. $\mathbf{v}=\langle 5,10\rangle$ e. $\mathbf{v}=\langle -\sqrt{10},-\sqrt{6}\rangle$ f. $\mathbf{v}=\langle 6\sqrt{2},\tfrac{1}{2}\rangle$

7. In the following, determine whether **u** and **v** are parallel, orthogonal, or neither.

a. $\mathbf{u}=4\mathbf{i}-6\mathbf{j},\ \mathbf{v}=-2\mathbf{i}+3\mathbf{j}$ b. $\mathbf{u}=2\mathbf{i}+18\mathbf{j},\ \mathbf{v}=\dfrac{3}{2}\mathbf{i}-\dfrac{1}{6}\mathbf{j}$ c. $\mathbf{u}=\langle \tan\theta,-1\rangle,\ \mathbf{v}=\langle \cos\theta,\sin\theta\rangle$

8. Write the vector **v** given its magnitude and the angle it makes with the positive *x*-axis.

a. $\|\mathbf{v}\|=8 \quad \theta=270°$ b. $\|\mathbf{v}\|=2 \quad \theta=60°$ c. $\|\mathbf{v}\|=12 \quad \theta=330°$

d. $\|\mathbf{v}\|=\dfrac{3}{8} \quad \theta=225°$ e. $\|\mathbf{v}\|=15 \quad \theta:$ direction of $2\mathbf{i}+3\mathbf{j}$ f. $\|\mathbf{v}\|=6$
θ: direction of $\mathbf{i}-6\mathbf{j}$

9. For each problem, draw a vector diagram and solve.

a. I push a power lawnmower with a force of 220 pounds with the handle at an angle of $20°$ to the ground. How much force is used in actually pushing the lawnmower forward?

b. A health-club exercise machine requires a user to pull down with force on a rope attached to pulleys with weights on the vertical as shown below. If the rope is at $65°$, how much weight will a person pushing down with a force of 250 pounds be able to lift?

c. If I swim at 2 miles an hour due north and a current of 5 mph is flowing due east, how fast do I swim and in what direction do I travel?

d. 2 men and a boy push a desk from behind it. Adam pushes with a force of 190 pounds straight ahead. Bill pushed with 150 pounds 12 degrees to the left of center and Charlie pushes with a force of 75 pounds 5 degrees to the right of center. What is the total force on the desk in moving it in the direction it travels and what direction relative to straight ahead does it go? (*Hint: draw a picture using either the x or y-axis as center*)

e. A motorboat crosses a river traveling at 14 mph with a current at 5 mph. What is the speed of the motorboat with the current and at what angle does it approach the opposite bank?

f. People in another motorboat wish to travel straight across the river that has a current of 5 mph. They set off at an angle θ to the bank and travel at 14 mph and go directly across the river (perpendicular to the current). At what angle θ did they travel to the riverbank and what was their resultant speed?

10. An object moves with position given by $\mathbf{r}(t) = (\sin t)(\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi$. a) Sketch the path of the object. b) The velocity of the object is $\mathbf{v}(t) = (\cos 2t)\mathbf{i} + (\cos t)\mathbf{j}$. Draw the velocity vectors on your graph at $t = 0, \pi/2, \pi$ and $3\pi/2$. c) sketch a graph of the speed function as a function of *t*.