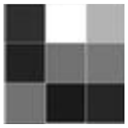
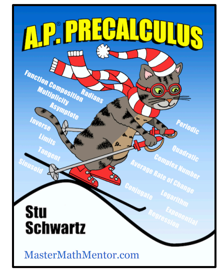
Topic 4.4 – Matrices and Notation * – Classwork





If you look at the very top left-most corner of a black and white picture on your cell phone, there are very tiny pixels – rectangular areas of shades of grey. If we convert this section to a value representing the grey-level intensity of the pixel with numbers close to 0 representing pixels that are very light and numbers close to 256 representing pixels that are close to black, we might get an array of numbers that looks like this: $\begin{bmatrix} 211 & 9 & 98 \\ 197 & 145 & 159 \\ 160 & 238 & 220 \end{bmatrix}$. This array is called a matrix. By adding 10 to each number, we could darken the picture. Other matrix operations can alter the image in other ways.

The material on matrices and determinants presented here serves an introduction to linear algebra. Linear algebra is a college level course usually taken after calculus, although it doesn't require calculus. Linear algebra is used in the natural sciences, business, economics, and in the social sciences Since methods involving matrices can require millions of numerical calculations, computers are very important in applying matrix techniques to a wide variety of practical problems. Your TI-calculator allows you to use matrices to do a number of mathematical process at once. The focus of this Precalculus course is the use of matrices and calculator use is encouraged to get answers quickly with no error. We make the assumption that students can define matrices on the calculator, enter the data, and add and subtract them. Page 72 of this unit gives a review of the basic keystro

In this unit, we will generally introduce each concept with a real-life application that shows how matrices applies to the problem. Most of the examples though will be done with matrices themselves without referencing an application. A section on matrix applications is given in chapter 4.

Suppose we are planning a Super Bowl party for our friends and we decide to have pizza and wings. We start by calling various pizza establishments and ask for prices for a large single-topping pizza, a two-liter container of soda, and an order of 10 wings. We can record this information in a table called a *matrix*.

	Pizza Hut	Dominos	Papa Johns	Vinny's
Pizza	$8.99	$7.59	$5.99	$12.59
Drinks	$2.59	$2.79	$2.39	$3.50
Wings (10)	$8.25	$7.95	$6.95	$11.49

$$\begin{bmatrix} 8.99 & 7.59 & 5.99 & 12.59 \\ 2.59 & 2.79 & 2.39 & 3.50 \\ 8.25 & 7.95 & 6.95 & 11.49 \end{bmatrix}$$

For simplicity sake, we can omit all the row and column headers as well as the dollar signs. We end up with a matrix as shown above right containing only numbers and we enclose it within brackets [ ].

Matrices are arrays of numbers arranged in rows and columns. This particular matrix has 3 rows and 4 columns. We say that the matrix has *order* or *dimension* 3 by 4 $(\text{written } 3\times 4)$. It is customary to say that this is a 3×4 matrix. Capital letters such as *A, B, C* are used to denote matrices and sometimes subscripts are used to define its dimensions. For example, the matrix above might be called $A_{3\times 4}$.

So, if $X = \begin{bmatrix} 5 & 3 & 8 \\ 3 & 0 & -2 \\ 6 & 9 & \frac{1}{3} \end{bmatrix}$ $Y = \begin{bmatrix} 2 & 8 \end{bmatrix}$ $Z = \begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix}$, they might be identified as $X_{3\times 3}$, $Y_{1\times 2}$, and $Z_{3\times 1}$. Since *X* has the same number of rows as columns, we call it a *square matrix*. If we want to identify a particular element of a matrix, we use the matrix name, row and column. So, in matrix *X*, the number 9, located in its 3rd row and 2nd column, would be identified as X_{32}. That is not the same as X_{23} which is –2.

Two matrices are *equal* is and only if they have the same dimensions and their corresponding elements are equal. Matrices of different dimensions can never be equal, even if they have the same elements, So,

$$\begin{bmatrix} 2 & 7 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and if } \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \text{ then } a = 1, b = 2.$$

The *zero matrix*, denoted $O_{m\times n}$ or just O, is the $m \times n$ matrix each of whose elements is 0. For example, some zero matrices are:

$$O_{1\times 3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The *negative* of the $m \times n$ matrix denoted by –*A* (read negative *A*), is the $m \times n$ matrix whose entries are the negatives of the corresponding entries in *A*.

If $A = \begin{bmatrix} -2 & 7 \end{bmatrix}$ then $-A = \begin{bmatrix} 2 & -7 \end{bmatrix}$. If $X = \begin{bmatrix} -5 & 0 \\ a-b & -3\pi \\ \sqrt{2} & e+2 \end{bmatrix}$ then $-X = \begin{bmatrix} 5 & 0 \\ b-a & 3\pi \\ -\sqrt{2} & -e-2 \end{bmatrix}$

Example 1) If $\begin{bmatrix} x-3 \\ y+4 \\ z-1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix}$, find *x*, *y*, *z*

Example 2) If $\begin{bmatrix} a & b \end{bmatrix} = M$ where $M = \begin{bmatrix} a^2 & \frac{1}{b} \end{bmatrix}$

How many possibilities are there for *M* ?

Example 3) If $\begin{bmatrix} 2x-3y \\ -6x+6y \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$, find *x*, *y*

Example 4) If $\begin{bmatrix} x+y & 3x-z \\ 2y-3z & x+y+z \end{bmatrix} = \begin{bmatrix} 3 & -10 \\ -2 & 7 \end{bmatrix}$

find *x*, *y*, and *z*

Going back to our Super Bowl party, suppose that it is decided that Vinny's is too expensive. It is also decided that rather than get drinks from the pizza place, to just buy them at cheaper prices at the supermarket. We then reduce our original matrix to $\begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix}$ where the rows now represent the price of a pizza and the price of 10 wings and the columns represent the prices at Pizza Hut, Dominos, and Papa Johns.

If we wanted to purchase 5 pizza and 5 orders of wings, we would represent this as $5\begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix}$, and we would multiply each matrix element by 5. $5\begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix} = \begin{bmatrix} 44.95 & 37.95 & 29.95 \\ 41.25 & 39.75 & 34.75 \end{bmatrix}$.

This process is *scalar multiplication*. If c is a real number (called a scalar) and A is an $m \times n$ matrix, then the product of c and A is denoted as cA, which is a matrix whose elements are the product of c and the corresponding elements of A.

Some results of scalar multiplication are these: $1A = A,\ OA = O,\ -1A = -A,\ cO = O$.

Example 5) Find $4\begin{bmatrix} -3 & 0 & -5 \\ \frac{1}{2} & \pi & -2 \\ -1 & \frac{-2}{3} & \sqrt{2} \end{bmatrix}$

Example 6) Solve for X: $4X = 3\begin{bmatrix} 8 & -20 & 0 \\ 2\pi & \sqrt{10} & -1 \end{bmatrix}$

Suppose we want to have our Super Bowl Party having toppings and dips with our pizza and wings. We contact Pizza Hut, Dominos, and Papa Johns and find the cost of toppings and dips. We then summarize this with a matrix as before.

	Pizza Hut	Dominos	Papa Johns
Each Topping	1.75	1.59	1.25
Each Dip	0.75	0.49	Free

$$\begin{bmatrix} 1.75 & 1.59 & 1.25 \\ 0.75 & 0.49 & 0 \end{bmatrix}$$

If we are interested in the cost for each establishment of a pizza with one topping and an order of wings with one dip, we can accomplish this by adding the corresponding components of our two prices matrices. If we let A be the basic price matrix and B be the added cost matrix, we can add A and B to get a third matrix C which represents the total price for a one-topping pizza and a one-dip order of wings at each establishment.

$$A = \begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix} \quad B = \begin{bmatrix} 1.75 & 1.59 & 1.25 \\ 0.75 & 0.49 & 0 \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} 8.99+1.75 & 7.59+1.59 & 5.99+1.25 \\ 8.25+0.75 & 7.95+0.49 & 6.95+0 \end{bmatrix} = \begin{bmatrix} 10.74 & 9.18 & 7.24 \\ 9.00 & 9.48 & 6.95 \end{bmatrix}$$

We define *matrix addition and subtraction*. If A and B are $m\times n$ matrices, then $C = A + B$ is a matrix whose sum or difference represents the sum or difference of the corresponding entries in matrices A and B. Symbolically, we have $C_{ij} = A_{ij} \pm B_{ij}$ where $1 \le i \le m$ and $1 \le j \le n$. It is therefore clear that we cannot add or subtract matrices of unlike dimensions. It also does not make sense to add matrices whose rows and column labels are not the same. For instance, if matrix B represented the cost of regular and premium fuel at 3 gas stations, while we could do the arithmetic in calculating $A + B$, there would be no meaning to the result. It's like adding 3 apples and 2 cars. We have 5 items, but that is about all.

There are several rules which can be proven concerning addition of matrices:
$A+B=B+A, A+(B+C)=(A+B)+C, A+0=0+A=A, A+(-A)=(-A)+A=0$

Because the sum of $A_{m\times n}$ and $-A_{m\times n}=O_{m\times n}$, we call $-A_{m\times n}$ the *additive inverse* of $A_{m\times n}$, For example, if $A=\begin{bmatrix} 2 & -5 \\ 0 & \frac{1}{4} \end{bmatrix}$, the additive inverse of A is $-A=\begin{bmatrix} -2 & 5 \\ 0 & \frac{-1}{4} \end{bmatrix}$ as $A+(-A)=\begin{bmatrix} 2-2 & -5+5 \\ 0+0 & \frac{1}{4}-\frac{1}{4} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Find the indicated sum or difference:

Example 7) $\begin{bmatrix} 5 & -1 & \frac{2}{3} \\ 0 & -4 & 3\pi \end{bmatrix}+\begin{bmatrix} 6 & 1 & \frac{-1}{2} \\ e & -9 & \pi \end{bmatrix}$

Example 8) $\begin{bmatrix} a+2 & b-3 \\ a^2-5 & b^2-a-b \end{bmatrix}-\begin{bmatrix} 1-a & 3-b \\ 5-a-a^2 & b-3a+b^2 \end{bmatrix}$

Find w, x, y, and z:

Example 9) $\begin{bmatrix} x+y & x-z \\ z+2 & w+z \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ 0 & 5 \end{bmatrix}$

Solve for X

Example 10) $\begin{bmatrix} 8 & -10 & 4 \\ -3 & 6 & 0 \end{bmatrix}-X=\begin{bmatrix} -5 & 10 & -1 \\ -2 & 6 & -6 \end{bmatrix}$

If $A=\begin{bmatrix} 4 & -1 \\ 6 & -5 \end{bmatrix}$ and $B=\begin{bmatrix} 8 & 15 \\ -6 & -10 \end{bmatrix}$, solve for X

Example 11) $X+4A=B$

Example 12) $\frac{1}{2}X-5A=2B$

Example 13) Solve the matrix equation

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix}-\begin{bmatrix} 4x \\ y \end{bmatrix}-6\begin{bmatrix} 2 \\ 1 \end{bmatrix}=O$$

Topic 4.4 – Matrices and Notation * – Homework

1. If $\begin{bmatrix} 2x-3 \\ 3y+4 \\ 1-\frac{2}{3}z \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 7 \end{bmatrix}$, find x, y, z

2. If $\begin{bmatrix} b & a \end{bmatrix} = \begin{bmatrix} \sqrt{a} & b+2 \end{bmatrix}$, find possible values of a and b

3. If $\begin{bmatrix} 5x+2y \\ 4x-3y \end{bmatrix} = \begin{bmatrix} 5 \\ 27 \end{bmatrix}$, find x, y

4. If $\begin{bmatrix} 2x+y & 3x+z \\ x-y-z & 3x-4y \end{bmatrix} = \begin{bmatrix} 4 & 19 \\ 25 & 72 \end{bmatrix}$ find x, y, and z

5. Solve for X

$$2X + 3\begin{bmatrix} 5 & 8 \\ -2 & 3 \\ 12 & -21 \end{bmatrix} = 4\begin{bmatrix} -1 & -5 \\ 0 & -3 \\ -6 & 7 \end{bmatrix}$$

6. Solve for X

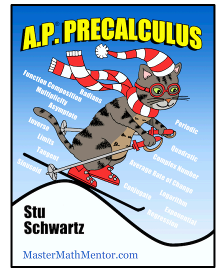
$$-10\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \frac{2}{3}X = 10\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} - X$$

If $A = \begin{bmatrix} 3 & \pi \\ \pi-4 & e^2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1-\pi \\ 3\pi+5 & 1-e^2 \end{bmatrix}$, find the additive inverse to

7. $A + B$

8. $5A - 2B$

Topic 4.5 –Matrix Operations – Classwork



When last we left our Super Bowl Party, we had established the price of a pizza (no toppings) and wings (no dips) at Pizza Hut, Domino's and Papa Johns with a cost matrix:

$$C=\begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix}.$$

Suppose we wanted to order 4 pizzas and 3 orders of wings. If we do the calculations without matrices, we multiply the pizza price by 4 and the wing price by 3 for each establishment.

Pizza Hut: $4(8.99)+3(8.25)=\$60.71$

Dominos: $4(7.59)+3(7.95)=\$54.21$

Papa Johns: $4(5.99)+3(5.95)=\$44.81$

To do this with matrices we establish a matrix N which represents the number of pizzas and wings we wish to order. This will be a 1×2 matrix. $\begin{matrix}\text{Pizzas} & \text{Wings}\\ 4 & 3\end{matrix} \Rightarrow N=\begin{bmatrix} 4 & 3 \end{bmatrix}$. When we multiply matrix N by matrix C, we expect the result to be another matrix whose elements give the total or purchasing 4 pizzas and 3 orders of wings at each of the 3 establishments. To do this, we multiply the row matrix N by each of the columns in the cost matrix C as follows:

$$N\times C=\begin{bmatrix} 4 & 3 \end{bmatrix}\times\begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix}$$

$$=\begin{bmatrix} 4(8.99)+3(8.25) & 4(7.59)+3(7.95) & 4(5.99)+3(6.95) \end{bmatrix}=\begin{bmatrix} 60.71 & 54.21 & 44.81 \end{bmatrix}$$

You can see from this example that matrix multiplication of this sort is defined only when the number of entries in the row matrix equals the number of rows in the multidimensional matrix. The product is a row matrix with the same number of entries as there are columns in the multidimensional matrix.

In general, matrix multiplication can only be calculated when a certain circumstance occurs. If a $(m\times n)$ matrix A is multiplied by a $(n\times r)$ matrix B, the result of $A\times B$ (or *AB)* will be a $(m\times r)$ matrix *P*. This process can be shown schematically on the right. If the number of columns in A is not the same as the number of rows in B, then $A\times B$ cannot be calculated. If they are, then the number of rows in the product $A\times B$ will be the same as the number of rows in A and the number of columns in the product $A\times B$ will be the same as the number of columns in B.

$$\begin{matrix} A & \times & B & = & P \\ (m\times n) & & (n\times r) & & (m\times r) \end{matrix}$$

[n's the same]

[Product uses the outside dimensions] ↗

To calculate the result of $A\times B$, every element in a row of A is multiplied by every element in a column of B. For instance, to multiply the 2×3 matrix A by the 3×2 matrix B, do the following:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\times\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \end{bmatrix}$$

Matrix multiplication can involve a lot of multiplication and addition and it is quite easy to make mistakes. Take it very slowly.

Example 1) If $A=\begin{bmatrix} 4 & -7 \\ 3 & 9 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 6 \\ -2 & 5 \end{bmatrix}$, find

a) AB b) BA c) A^2

Example 2) Find the product of $\begin{bmatrix} 5 & 4 & -1 \\ -3 & -1 & 10 \end{bmatrix} \times \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ -2 & 8 \end{bmatrix}$

By this example, it should be clear that in general, the product of matrices AB and BA are not necessarily commutative. That is, AB is not necessarily the same as BA. And that is only when it is possible to take a product. It is quite possible that AB can be calculated and BA not be calculated. The only time that both AB and BA can both be taken is when A and B are both square matrices.

The calculator can multiply an $m \times n$ matrix by an $n \times r$ matrix. If matrix A and matrix B, both 2×2 matrices are input, here are the keystrokes to find AB. We will generally use the calculator because it is quicker and doesn't make errors unless input matrices are not correct.

```
[A][B]
        [ [22  -1]
          [-32 80] ]
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Example 3) In our Super Bowl example, suppose we were interested in getting either 4 pizzas and 3 orders of wings or 6 pizzas and 2 orders of wings and we wish to price them. Show how we can do so with matrices:

$$N \times C = \begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix} \times \begin{bmatrix} 8.99 & 7.59 & 5.99 \\ 8.25 & 7.95 & 6.95 \end{bmatrix} = \begin{bmatrix} 60.71 & 54.21 & 44.81 \\ 70.44 & 61.44 & 49.84 \end{bmatrix}$$

4 pizzas and 3 wings at Pizza Hut is $60.71, at Dominos is $54.21 and at Papa Johns is $44.81

6 pizzas and 2 wings at Pizza Hut is $70.44, at Dominos is $61.44 and at Papa Johns is $49.84

Example 4) Find the following by hand and confirm using the calculator.

a) $\begin{bmatrix} 4 & 6 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 7 & -6 \\ 2 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 5 & 3 & -8 \\ -2 & -1 & 7 \end{bmatrix}\begin{bmatrix} -6 & -5 & 0 \\ 1 & 3 & -4 \\ 4 & 9 & -2 \end{bmatrix}$

Example 5) Solve for the variables:

a) $\begin{bmatrix} -5 & 0 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 10 \\ -8 \end{bmatrix}$

b) $\begin{bmatrix} 4 & -6 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -20 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 4 & -4 \\ 5 & 8 \end{bmatrix}=\begin{bmatrix} 30 & 22 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 3 \\ -7 \end{bmatrix}$

Example 6) Let matrix $A=\begin{bmatrix} 4 & 6 \\ 16 & 28 \end{bmatrix}$ represent the fact that on a river boat cruise ship, regular staterooms require 4 rolls of toilet paper and 16 light bulbs. Suites require 6 rolls of toilet paper and 28 light bulbs. Let matrix $B=\begin{bmatrix} 8 & 12 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ represent the fact that there are 3 decks with the lowest deck having 8 regular rooms and 2 suites, the middle deck with 12 regular rooms and 4 suites, and the highest deck with 2 regular rooms and 6 suites. Determine the product *AB* and tell what its entries represent.

The Identity Matrix

As stated, matrix multiplication is not commutative. However, there are several matrix products that do not depend on the order of the factors. One of them is called the *identity matrix*. For a 2×2 matrix, the identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Just as the number 1 is called the multiplicative identity because for any real number x, $x(1) = 1(x) = x$, for any 2×2 matrix $A \neq 0$, $AI = IA = A$.

Example 7) For each given matrix M, shown that $MI = IM = M$

a) $\begin{bmatrix} 8 & -2 \\ -3 & \frac{1}{2} \end{bmatrix}$

b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

In general, the identity matrix for any square matrix is given by $I = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix}$.

Determinants of 2x2 Matrices

In linear algebra, every square matrix has a determinant associated with it. This is a single number that is helpful in solving systems of linear equations (simultaneous equations). You probably studied them in algebra 2. The determinant of a matrix A written as $|A|$. When we take the absolute value using $| \quad |$ notation , answers are always positive. But that notation when referring to a matrix means the determinant which can be negative, For 2×2 matrices, the value of the determinant of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

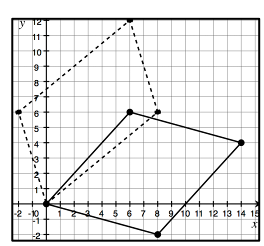
Example 8) Find the value of the determinant of following matrices:

a) $\begin{bmatrix} 8 & 6 \\ -2 & 6 \end{bmatrix}$

b) $\begin{bmatrix} \sqrt{6} & \frac{-8}{5} \\ -\frac{2}{3} & \sqrt{6} \end{bmatrix}$

c) $\begin{bmatrix} 8 & 2 \\ -3 & \frac{-3}{4} \end{bmatrix}$

The value of the determinant is the area of the parallelogram formed by the row or column vectors. In 8a) the solid lines represent the parallelogram formed by the vectors $\langle 8,-2 \rangle$ and $\langle 6,6 \rangle$ while the dashed lines represent the parallelogram formed by the vectors $\langle 8,6 \rangle$ and $\langle -2,6 \rangle$. Each has area as the value of the determinant, 60. If the value of the determinant is zero as in c), the row and column vectors are parallel.



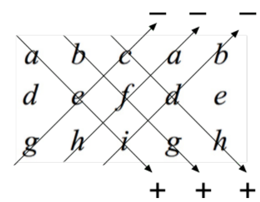
Determinants of 3x3 Matrices *

For 3×3 matrices, the formula is much more complicated:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + cdh + bfg - afh - bdi - ceg$$

That is far too complicated to remember. However, there are several ways to compute it without that messy formula:

First, write the 3 by 3 determinant and then rewrite the first and 2nd column to the right of it. Draw the diagonals as shown. Add the products of the descending diagonals to the negative of the products of the ascending products:



$$aei + bfg + cdh - ceg - afh - bdi$$

A second way is to use the concept of *minor determinants*. We take each element on the first row, alternating signs, and multiply each by the 2×2 minor matrix below it not using that column. It is easier to show it rather than explain it.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$a(ei - fh) - b(di - fg) + c(dh - eg) = aei + bfg + cdh - afh - bdi - ceg$$

This method is easily remembered and is the one recommended for calculation by hand. We will see that it can be extended to larger square matrices as well. However, if you need to calculate the value of a determinant of a matrix larger than 2×2, it is wise to use a calculator that has matrix and determinant capabilities.

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Example 9) Find the value of the determinants of the following matrices.

a) $\begin{bmatrix} 2 & 3 & -4 \\ 5 & 1 & 0 \\ -6 & -2 & 7 \end{bmatrix}$

b) $\begin{bmatrix} a & b & -b \\ b & a & a+b \\ 2 & 3 & -2 \end{bmatrix}$

The TI-calculator has the ability to take determinants of matrices that are input. This is found in the

[2nd] [Matrix] [MATH] [1) det(] menu. Let [A] be defined as in example 9a above.

```
Det([A])
          -75
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Inverses

Every non-zero real number has a multiplicative inverse. That is, for all *n*, $n \neq 0 \Rightarrow n\left(\frac{1}{n}\right) = \frac{1}{n}(n) = 1$. So, the multiplicative inverse of 2 is ½ and the multiplicative inverse of –2/3 is –3/2. We symbolize the multiplicative inverse for *n* as n^{-1} . Since *n* is a number, $n^{-1} = \frac{1}{n}$.

Is this true for matrices? For instance, does every non-zero 2×2 matrix have a multiplicative inverse? If they do, then for all non-zero matrices *A*, $AA^{-1} = A^{-1}A = I$. In this case, A^{-1} does not mean $\frac{1}{A}$ because we cannot divide scalars by matrices. A^{-1} must be another 2×2 matrix.

We can readily see that the matrix $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ has a multiplicative inverse $A^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ because

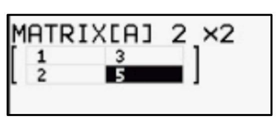
$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

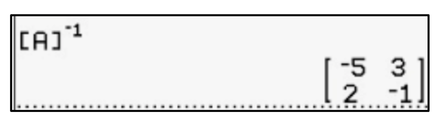
We will not prove it, but the inverse of the non-zero 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $|A|$ is the determinant of matrix A . So, to find A^{-1}, interchange *a* and *d*, replace *b* and *c* by their negatives and multiply by the reciprocal of the determinant of *A*. If $|A| = 0$, then the matrix of *A* has no inverse. Conversely, if $|A| \neq 0$, then *A* has an invese.

Example 10) Find the inverse of the following matrices:

a) $\begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix}$ c) $\begin{bmatrix} -8 & 10 \\ 6 & 5 \end{bmatrix}$

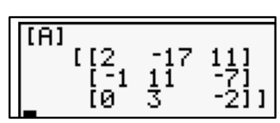
The TI-calculator has the ability to take inverses of matrices that are input. On the home screen , you can get matrix *A* by the keystrokes [2nd] [Matrix] [1:[A]] .

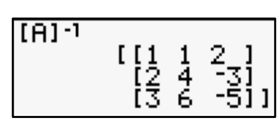




Then hit the reciprocal x^{-1} key. The calculator knows that this is not a reciprocal but an inverse when applied to matrices.

We will not go into the process of finding inverses for square matrices 3 by 3 or greater as it is cumbersome. But the calculator's ability to do this make it easy to determine. For instance:





Matrix equations in the form of $AX = B$ may be solved using inverse matrices.

$AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$. Since matrix multiplication is not commutative, be careful to write the solution as $X = A^{-1}B$, *not* $X = BA^{-1}$.

Example 11) Solve the following equations for the matrix X: Calculator permitted.

a) $\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} X = \begin{bmatrix} 4 & -3 \\ 7 & -4 \end{bmatrix}$

b) $\begin{bmatrix} 20 & 12 \\ -16 & 8 \end{bmatrix} X = \begin{bmatrix} 40 & 344 \\ 320 & -240 \end{bmatrix}$

This type of problem is important as we can write a system of equations: $\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}$ using matrices:

$$\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix} \Rightarrow \begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ So } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Example 12) Solve the system of equations: $\begin{matrix} 5x - 3y = 35 \\ 6x + 5y = -1 \end{matrix}$

Example 13) Use your calculator to solve $\begin{matrix} 5x - 4y + 4z = 8 \\ 3x + 6y - 5z = -27 \\ 8x + 2y + 3z = 5 \end{matrix}$

Example 14) Use your calculator to solve $\begin{matrix} 2w + 4x - 3z = 10 \\ 3w + 8y + z = 7 \\ -3x + 5z = -11 \\ x + y + 2y + z = -3 \end{matrix}$

Topic 4.5 – Matrix Operations – Homework

1. If $A=\begin{bmatrix} -1 & 4 \\ -2 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -4 \\ 5 & -2 \end{bmatrix}$, find the following. Confirm by calculator.

   a. AB b. BA c. A^2

2. Using the calculator, find the following.

   a. $\begin{bmatrix} 8 & 0 & -1 \\ -2 & 6 & -10 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -3 & 4 \\ -1 & 0 \end{bmatrix}$

   b. $\begin{bmatrix} 0.5 & 0.4 \\ 0.2 & 0.3 \\ -0.1 & 0.5 \\ 0.8 & -0.4 \end{bmatrix}\begin{bmatrix} 0.6 & 0.6 & -0.3 \\ -0.2 & 0.1 & 0.9 \end{bmatrix}$

3. Solve for x and y: $\begin{bmatrix} 3 & -8 \\ 11 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -46 \\ -59 \end{bmatrix}$

4. Let matrix $A=\begin{bmatrix} 12 & 16 \\ 3 & 5 \end{bmatrix}$ represent the fact a company makes two types of business jets. The smaller one is named the Challenger and seats 12 and has 3 tires. The larger jet is named the Phantom and seats 16 and has 5 tires. Let matrix $B=\begin{bmatrix} 4 & 6 & 0 \\ 3 & 5 & 8 \end{bmatrix}$ represent the fact that there are 3 factories for these planes. One factory can simultaneously build 4 Challengers and 3 Phantoms. Another factory can simultaneously build 7 Challengers and 5 Phantoms. The third factory only builds Phantoms and can build 8 of them simultaneously. Determine the product AB and tell what its entries represent.

5. Let the matrix $A = \begin{bmatrix} 24 & 15 \end{bmatrix}$ represent the fact that a custom furniture company produces 24 sofas and 15 love seats each week. Let the matrix $B = \begin{bmatrix} 3 & 7 \\ 2 & 5.5 \end{bmatrix}$ represent the fact that a sofa requires 3 hours for construction and 7 hours for upholstery while a loveseat requires 2 hours for construction and 5.5 hours for upholstery. Let the matrix $C = \begin{bmatrix} 12.25 \\ 18.50 \end{bmatrix}$ represent the fact that the labor costs per hour are \$12.25 for construction and \$18.50 per hour for upholstery. Find the product ABC and specify what its entry represents.

6. Find the value of the determinant of the given matrix. Confirm by calculator.

a. $\begin{bmatrix} -9 & -11 \\ 6 & -8 \end{bmatrix}$

b. $\begin{bmatrix} 2\sqrt{12} & \frac{-4}{\sqrt{3}} \\ \frac{-2}{\sqrt{12}} & 4\sqrt{3} \end{bmatrix}$

c. $\begin{bmatrix} a-b & a\sqrt{a-b} \\ b\sqrt{a-b} & (b-a)^2 \end{bmatrix}$

7. Find the value of each inverse matrix

a. $\begin{bmatrix} 8 & 7 \\ -2 & -2 \end{bmatrix}^{-1}$

b. $\begin{bmatrix} 15 & -20 \\ -15 & 25 \end{bmatrix}^{-1}$

c. $\begin{bmatrix} a & a-b \\ b & b-a \end{bmatrix}^{-1}$

8. Solve the system of equations. Calculators permitted.

a. $$\begin{aligned} 8x - 9y &= 19 \\ -5x + 6y &= -13 \end{aligned}$$

b. $$\begin{aligned} 5x - 3y - 4z + 46 &= 0 \\ 8x + 9y + 3z + 19 &= 0 \\ -7x - 2y + 5z - 29 &= 0 \end{aligned}$$

