Slip-Sliding Away

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I was watching a major league baseball game the other night and I saw a play that got me thinking about calculus. A batter hit a slow roller towards 3rd base. He was quite fast. As he neared the 1st base bag, he went into a slide. He was called safe but the play was video-reviewed and instant replay showed that he was out. The ball just beat him to the bag. The commentator remarked that if he had continued to run through the bag and not gone into a slide, he would have been safe. That got me to thinking about the problem from a calculus point of view.

First, for people who are not that much into baseball, here are a few explanations. When runners go into 2nd base on a close play, they usually slide. That is because that they have to maintain contact with the bag. If they overrun the base (as does happen), they will be tagged out. The slide has several benefits. First, it brings the runner low and since throws to the base are usually higher up, it takes time for the 2nd baseman to bring the ball down to tag the runner. We are talking fractions of a second and that can make a difference between being called out and called safe. And second, going into a slide slows the runner somewhat and allows his hands to be at ground level in order for him to grab the bag and hold onto it, just as a boater will grab a buoy to slow down and stop.

The same argument is true for 3rd base.

Home plate is trickier as a runner simply needs to touch home plate before he is tagged but doesn't have to maintain contact with it. Still, most players slide into home because the throw to home is usually high and the catcher has to bring the ball down, taking time. Also, any part of the runner's body touching home is allowed and runners can sometimes avoid tags by using their hands or feet.

This problem is about 1st base. The rules of baseball state that a runner is allowed to overrun 1st base. If a runner had to maintain contact with the bag, it would be necessary to slow down while approaching it. This would give the infielders more time to catch and throw the ball and there would be a great deal less offense in a game. It would be boring.

So it is to his benefit for a runner to be moving as fast as he can throughout the entire run to 1st base. However, it is permissible for a runner to slide into 1st base as well if he so wants to. But is it beneficial?

The rules of baseball say that the 1st baseman simply has to have contact with the bag when catching the ball and the runner is out if the ball is caught before the runner hits the bag. Sometimes though, if a throw is wayward, a 1st baseman will be forced to tag the runner and if the runner see this, sliding may make tagging difficult. This is exactly what is happening in the picture to the left.

This problem is just concerned with the runner who slides into 1st base rather than running through the bag because he thinks he can get there just a bit quicker. This action usually isn't planned. Instinct tells the runner that be might get there quicker by sliding. Let's suppose you are the one running and when you get close to 1st base, you decide to slide.

As you are running down the line, you have a certain velocity. That velocity is provided by your legs.

At the moment you go into your dive, your velocity is affected in two ways:

- Your legs are no longer propelling you so you must lose a little velocity.
- However, as you push off the balls of your feet, and subsequently fly through the air towards impact on the ground, there is less friction than there was with your feet hitting the ground so you gain a little velocity.

Realize though that this gain in velocity is just for a split second. Like a bullet leaving a gun, the expanding gas behind the bullet propels the bullet down the barrel of a gun. Once it leaves the barrel, it does not gain velocity. It is at its fastest the moment it leaves the barrel.

Of course, when you actually hit the ground, there is major friction taking place and you quickly slow down.

The question on the table: is that slight loss of velocity because of your legs no longer providing it offset by the gain in velocity because of lack of friction while flying through the air?

All this translates to the question: *should you slide into 1st base*?

With all of that said, let's look at a problem, contrived though it is, based on some realistic data.

Suppose a runner is running to 1st base. Let's make the assumption that he is running at a constant rate of speed from the outset: 30 feet per second. Of course, this cannot be true, as after he hits the ball, he starts at zero velocity and must get up to speed. However to make the numbers slightly easier to work with, let's assume that his speed is constant to 1st base. 30 feet per second is realistic for the faster runners in the major leagues.

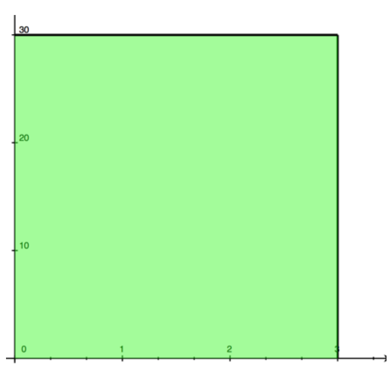
Since the distance from home plate to 1st base is 90 feet, we know it will take the runner 3 seconds to get from home to 1st. And because the velocity is constant throughout, the acceleration is zero.

Let's assume that our runner decides to go into a slide at $t = 2\frac{2}{3}$ seconds. At $t = 2.70$ seconds, let's assume that his acceleration has increased just a hair: $a = 0.5$ feet per second^2. And at $t = 2.80$ seconds, he is already on the ground and slowing and his acceleration is $a = -0.5$ feet per second^2.

The question is: Given that information, is sliding beneficial? Meaning, does he get to the bag quicker if he slides than if he continued to run?

Here is the solution:

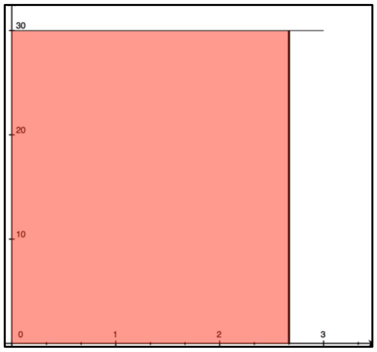
Ultimately this is a calculus question, so let's use calculus from the outset. We know that distance is the area under the velocity curve. We start with the runner running the entire distance at 30 feet/sec. In 3 seconds, the distance he runs is given by



$$D = \int_0^3 v(t)\,dt = \int_0^3 30\,dt = 30t\Big]_0^3 = 90 \text{ feet}$$

Of course, calculus is not needed here. The distance run is simply the area of a rectangle (base • height).

Now, let’s assume that our runner intends to be in the act of sliding the last 10 feet. So he will run 80 feet and then start to slide. Let’s find out how long he runs before starting to slide.



$$\int_0^k v(t)\,dt = 80$$

$$\int_0^k 30\,dt = 30t\Big]_0^k = 80$$

$$30k = 80 \Rightarrow k = \frac{80}{30} = 2\tfrac{2}{3} \text{ seconds}$$

For the acceleration a, we know three points in the form of (t, a): $\left(2\tfrac{2}{3},0\right),\left(2.7,0.5\right)$, and $\left(2.8,-0.5\right)$. 3 points will determine a parabola. We do not know that the acceleration is in this shape but since there is a very small-time duration between $t = 2\tfrac{2}{3}$ and $t = 2.8$, the actual curve will not matter that much. Let’s generate the equation of that parabola using those 3 points and simultaneous equations. There are many ways to solve the simultaneous equations but let us use matrices to do so. Recall that matrix A represents coefficients of A, B and C in our equations and matrix B represent the constants on the other side of the equation, the solution to the simultaneous equations is $A^{-1} \bullet B$. We use the calculator to do the matrix multiplication.

$$a = At^2 + Bt + C$$

$$\begin{aligned}
\left(2\tfrac{2}{3},0\right)&: 0 = 7.111A + 2.667B + C \\
\left(2.7,0.5\right)&: 0.5 = 7.29A + 2.7B + C \\
\left(2.8,-0.5\right)&: -0.5 = 7.84A + 2.8B + C
\end{aligned}
\qquad
\begin{bmatrix} 7.111 & 2.667 & 1 \\ 7.29 & 2.7 & 1 \\ 7.84 & 2.8 & 1 \end{bmatrix}^{-1}
\begin{bmatrix} 0 \\ 0.5 \\ -0.5 \end{bmatrix}
=
\begin{bmatrix} -187.5 \\ 1021.25 \\ -1390 \end{bmatrix}$$

$$a = -187.5t^2 + 1021.25t - 1390,\ t > 2\tfrac{2}{3}$$

Now that we know $a(t)$ for $t > 2\tfrac{2}{3}$, let us calculate $v(t)$ by integrating a. When we integrate, we get a constant of integration C. We know that at $t = 2\tfrac{2}{3}, v = 30$. So we need to use our point $\left(2\tfrac{2}{3},30\right)$ to find the value of C.

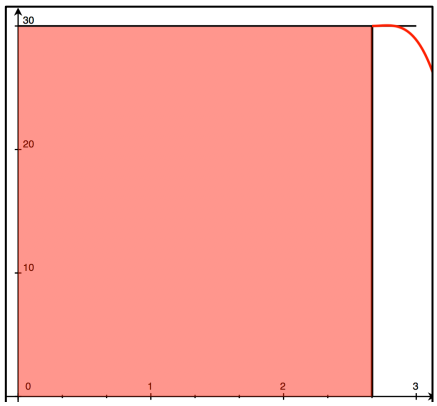
$$v(t) = \int a(t)\,dt = \int\left(-187.5t^2 + 1021.25t - 1390\right)dt = -62.5t^3 + 510.625t^2 - 1390t + C$$

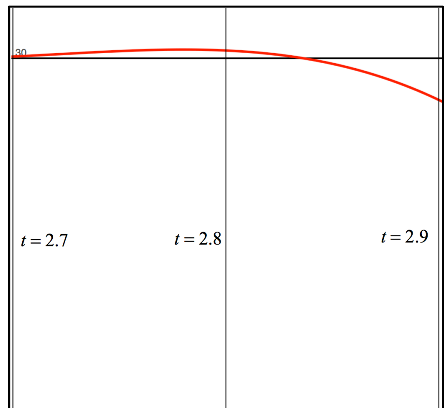
$$t = 2\tfrac{2}{3}: 30 = -62.5\left(\frac{8}{3}\right)^3 + 510.625\left(\frac{8}{3}\right)^2 - 1390\left(\frac{8}{3}\right) + C$$

$$30 = -1260.741 + C \Rightarrow C = 1290.741$$

$$v = -62.5t^3 + 510.625t^2 - 1390t + 1290.741,\ t > 2\tfrac{2}{3}$$

We now graph this velocity for $t > 2\frac{2}{3}$. If you look at it very carefully, you will see it rise just slightly above 30. This is because at the moment when the slide began, the runner pushed off his foot giving himself some extra impetus which we saw by the acceleration being 1 at $t = 2.7$. However then it goes down quickly when the runner no longer has his legs to provide velocity and descends dramatically when he hits the ground and friction slows him down. To the right, we zoom into that point to see the slight increase in velocity.





It is now easy to show that sliding in this way doesn't get the runner to the 1st base bag faster. We showed previously that if the runner had run straight through the bag, in that last 1/3 of a second, he would have covered 10 feet as $\frac{1}{3}(30) = 10$. Let us see how much distance the runner who slid would have covered in the same time frame. Since v is a polynomial, we can integrate easily. However, it is just easier to use the calculator to find this value.

$$d = \int_{8/3}^{3} \left(-62.5t^3 + 510.625t^2 - 1390t + 1290.741\right) dt$$

$$d = \left[\frac{-62.5t^4}{4} + \frac{510.625t^3}{3} - 695t^2 + 1290.741t\right]_{8/3}^{3} = 9.938 \text{ feet}$$

It should be apparent then that sliding is just a hair slower than running through the bag as the runner covers 0.062 more feet in that 1/3 of a second that the slider. However, if we would like to determine exactly how long it will take the runner to get to the bag by sliding, we need to solve the following equation for k.

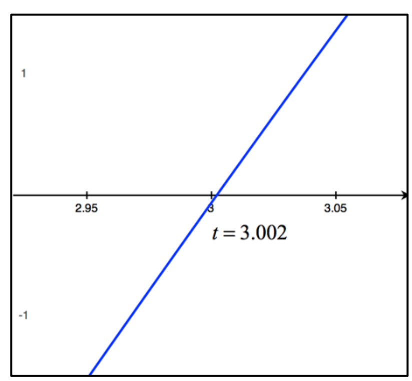
$$\int_{8/3}^{k} \left(-62.5t^3 + 510.625t^2 - 1390t + 1290.741\right) dt = 10$$

$$\left[\frac{-62.5t^4}{4} + \frac{510.625t^3}{3} - 695t^2 + 1290.741t\right]_{8/3}^{k} = 10$$

$$\frac{-62.5k^4}{4} + \frac{510.625k^3}{3} - 695k^2 + 1290.741k - 937.285 = 10$$

$$\frac{-62.5k^4}{4} + \frac{510.625k^3}{3} - 695k^2 + 1290.741k - 947.285 = 0$$

Graphing, we get $k = 3.002$ seconds



The commentator was probably correct in saying that if the runner hadn't slid, he would have been safe. That 0.062 feet is about ¾ of an inch. Video replay showed the distance between the runner's hand and the 1st base bag at the moment that the ball was in the glove was about ½ inch.

The following thought may have crossed your mind. In the graph of our velocity for $t > 2\frac{2}{3}$, the curve goes slightly above the line $v = 30$. Suppose the slide is delayed for 1-tenth or even 2-tenths of a second. Would our slightly increased velocity ensure the runner will get to the bag quicker?

That would be true if he were playing football where the goal to score a touchdown is simply to have the ball break the plane of the goal line. Players propel themselves through the air all the time to avoid being tackled by players who are close to the ground. Also, in football, how fast a player gets into the end zone is not important. It doesn't matter if he scores a half-second later.

In running, runners are actually taught by some coaches to lunge for the tape. In the 2016 Olympics, 400-meter Sprint event, Allyson Felix lost a gold medal to Shanaue Miller because Miller dove at the end. In big running events like the Olympics, tapes are no longer used. Video replay determines who crosses the line first, meaning the torso, which is different from head or arms, crosses the finish line. That is why runners lean forward in close races – to get their torsos across the line.

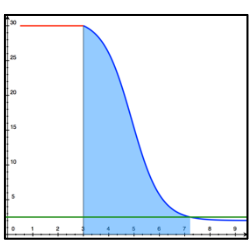
But diving in races is legal. But, as ESPN's Sport Science feature explains, what matters is when you start the dive. As we have seen, the start of the dive is actually faster — for that split second — than running, but the moment your feet leave the ground on a dive, deceleration occurs. If a track and field runner dives at the wrong moment, it could easily mean losing. And it can result in an injury.

But in baseball, the runner has to touch the bag. Delaying his slide would mean that he may still be in the air at the time he crosses the plane of the bag. If he doesn't actually make contact with the bag, he is out. And he has to be on the ground to do that. It may be possible to time the start of the slide so that he can still be in the air and reach down and lightly swipe the bag, but we are talking fractions of a tenth of a second. No baseball player can be that precise, especially given the speed that events occur.

One more interesting piece. Most of the time, our player runs through the bag. But once they hit the 1st base bag, they slow down. But the slow-down is gradual as to not injure themselves. Some go quite far down the 1st base line into right field. Let's model that.

Suppose the velocity once the runner hits the bag at $t = 3$ seconds is given by $v = 31 - \dfrac{29}{1+5350e^{-1.75t}}$. When the runner's speed is 2.5 feet per second, he turns around and walks back to the base. How far up the line did he go?

Solution: We graph the new velocity function as well as $v = 2.5$ and find that their intersections is when $t = 7.216$ seconds as shown. We then integrate the velocity function using the calculator.



$$d = \int_3^{7.216} v(t)\,dt =$$

$$\int_3^{7.216} \left[31 - \frac{29}{1+5350e^{-1.75t}}\right] dt = 63.987 \text{ feet}$$

We will study the shape of this graph in BC calculus. It is called a logistic curve. Obviously the runner's speed is still fast at the moment he hits the bag and decreases very quickly. As he gets closer to stopping though, his speed decreases much more slowly. He wants to stop gradually. Injuries occur when sudden moves are made.

On the following page, there is a sample AP problem that uses this running versus sliding theme with numbers that are easier to use. The solution follows. Feel free to use it with your students.

Sample AP Problem on Straight-Line Motion

A baseball player hits a ground ball and sprints to 1st base. For the first 0.5 seconds, his velocity is given by $v(t) = 120t - 120t^2$ where t is measured in seconds and v is measured in feet. The runner hits his maximum speed of 30 ft/sec at $t = 0.5$ and continues at that speed until he touches 1st base.

a) Write v as a piecewise function and show that v is differentiable. **(2)**

b) The distance from home plate to 1st base is 90 feet. Determine how long it will take the runner to reach 1st base. Show the reasoning that leads to your answer. **(2)**

c) Suppose the runner begins to slide at $t = 3$ seconds. His acceleration for $t > 3$ is given by $a(t) = -75t^2 + 450t - 675$. Find his velocity for the time that he is sliding. **(2)**

d) Give an argument that shows why sliding into 1st base in this way isn't as fast as continuing to run. **(3)**

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a) Write v as a piecewise function and show that v is differentiable. **(2)**

$$v(t)=\begin{cases}120t-120t^2, t\le 0.5\\ 30, t>0.5\end{cases}$$ Both pieces of v are differentiable

$\lim_{t\to 0.5^-} v(t)=120(0.5)-120(0.25)=60-30=30 \quad \lim_{t\to 0.5^+} v(t)=30$

So v is continuous

$$v'(t)=\begin{cases}120-240t, t\le 0.5\\ 0, t>0.5\end{cases}$$

$\lim_{t\to 0.5^-} v'(t)=120-240(0.5)=120-120=0 \quad \lim_{t\to 0.5^+} v'(t)=0$

So v is differentiable at $t=0.5$ and thus differentiable everywhere

1 pt for writing v and showing continuity
1 pt for showing v is differentiable

b) The distance from home plate to 1st base is 90 feet. Determine how long it will take the runner to reach 1st base. Show the reasoning that leads to your answer. **(2)**

Distance reached in the first 0.5 seconds:

$$\int_0^{0.5}\left(120t-120t^2\right)dt=60t^2-40t^3\Big]_0^{0.5}=60\left(\frac{1}{4}\right)-40\left(\frac{1}{8}\right)=10 \text{ feet}$$

The remaining 80 feet takes $30k$ seconds

$30k=80\Rightarrow k=\frac{8}{3}$ seconds

So it takes $\frac{1}{2}+\frac{8}{3}=\frac{19}{6}$ or 3.167 seconds

1 pt for distance in first 0.5 second
1 pt for analysis and
answer for remaining 80 feet

c) Suppose the runner begins to slide at $t = 3$ seconds. His acceleration for $t > 3$ is given by $a(t)=-75t^2+450t-675$. Find his velocity for the time that he is sliding. **(2)**

$v(t)=\int\left(-75t^2+450t-675\right)dt=-25t^3+225t^2-675t+C$

$v(3)=30=-25(27)+225(9)-675(3)+C$

$30=-675+C\Rightarrow C=705$

$v(t)=-25t^3+225t^2-675t+705$

1 pt for correct integration
1 pt for finding C and correct v

d) Give an argument that shows why sliding into 1st base in this way isn't as fast as continuing to run. **(3)**

Running last $1/6$ sec	Sliding last $1/6$ sec
$d=30\left(\frac{1}{6}\right)=5$ feet	$d=\int_3^{19/6}\left(-25t^3+225t^2-675t+705\right)dt=4.995$ feet

Alternately: $v'(t)=a(t)=-75\left(t^2-6t+9\right)=-75(t-3)^2=0$

v is maximized at $t=3$. When $t>3, v'(t)<0$ so v is decreasing

So the distance traveled must be less than it would if $v=30$

1 pt for comparing distance on $\left[3,\frac{19}{6}\right]$
1 pt for finding running distance
1 pt for finding running distance

or

1 pt for $v'(t)$
1 pt for v always decreasing
1 pt for explanation