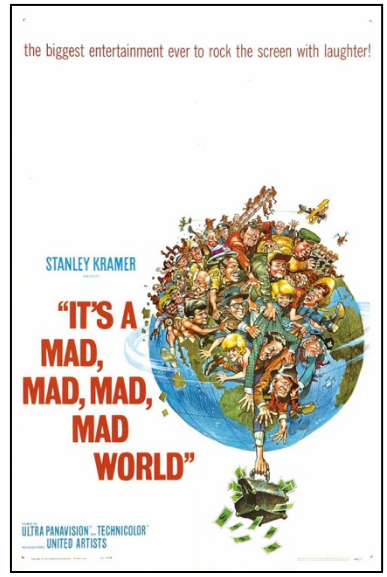
What’s Fair is Fair …

This is an article about fairness. What exactly does “fairness” mean?



When I was a kid, a movie came out in 1963 called “It’s a Mad, Mad, Mad, Mad World.” This film, directed by Stanley Kramer, was a zany, madcap movie that ran close to 4 hours and included practically every Hollywood star at the time, young and old. The plot is much too convoluted to tell here, but I would like to begin with the very start of the movie.

“Smiler” Grogan, a just-released convict jailed for robbery 15 years earlier, escapes police surveillance and is speeding dangerously on a mountainous California highway and flies right off a cliff. Five motorists stop to help him: Just before he dies, Grogan tells them about $350,000 buried in Santa Rosita State Park under “a big W.” After failing to come up with a satisfactory way to split the money, a race begins to find it first. “Mad World’ is a comedy about American greed.

Early in the movie, the question becomes that if the people work together and are able to find the $350,000, how should it be divided fairly. Everyone has a different opinion. Rather than me spell it out, why not watch the clip? The link is below but if you do not want to type it, just Google mad mad world equal shares and you will see it first. The clip is 5 minutes 41 seconds and is well worth watching to understand the issue and enjoy the humor from these legendary actors and actresses.

https://www.youtube.com/watch?v=sXMyXypjmGM

Let’s summarize what is on the video.

The people involved are: Melville Crump, a dentist on a second honeymoon with his wife Monica; Lennie Pike, a furniture mover; Ding Bell and Benjy Benjamin, two friends on their way to Las Vegas; and J. Russell Finch, a seaweed-business owner, traveling with his wife Emmeline and his loud, obnoxious mother-in-law, Mrs. Marcus.

There were 4 cars that stopped, and 5 people went down to the wreck: Melville Crump, Lennie Pike, Ding Bell, Benjy Benjamin and Russell Finch. Since this is the 60’s, the women stayed at the cars. This can be summarized using the table to the right.

Cars	People At Scene	People in Cars
1	Melville Crump	Melville Crump Monica Crump
2	Lennie Pike	Lennie Pike
3	Ding Bell Benji Benjamin	Ding Bell Benji Benjamin
4	Russell Finch	Russell Finch Emmeline Marcus Mrs. Marcus

After Grogan figuratively and literally "kicks the bucket", they all meet at the cars. Mr. Crump suggested that they all calmly and safely drive the 200 miles to Santa Rosita State park, hunt for the money and if they find it to split it with "fair shares for everyone." And that is where the argument started.

Loud Mrs. Marcus questions "what's this fair shares for everyone mean?" Mr. Crump suggested that since there were 4 cars, then the treasure, if found, should be split 4 ways "in quarters." However, immediately there was dissension. Ding Bell felt that this was grossly unfair. He immediately thought of only the 4 people involved, not considering that they would share it with their family. His point was that each of the 3 car owners (which he assumed to be the men) would receive $87,500 while he and Benji Benjamin would have to split that sum, each only getting $43,750. Still, we know that these people were together and would be participating in the windfall.

Cars	Car Allocation	People in Cars	Each Gets
1	87,500	Melville Crump	87,500
		Monica Crump	-
2	87,500	Lennie Pike	87,500
3	87,500	Ding Bell	43,750
		Benji Benjamin	43,750
4	87,500	Russell Finch	87,500
		Emmeline Marcus	-
		Mrs. Marcus	-
Total	350,000		350,000

The Crump family (assuming that they were on good terms) would be sharing $87,500. Obviously, Lennie Pike, as a singleton, would be getting $87,500. And the Finch family (assuming that they were all on good terms, which they were not) would be sharing $87,500. But Ding and Benji were guys just traveling together and they would have to share the $87,500 equally, each getting $43,750. It is clear though that Mr. Pike in green as an individual fares the best. And if the Finch Family doesn't stay together, then they would be forced to share the money 3 ways. So Mr. Finch proposes to split the money based on people at the scene.

There were 5 people who went down to the wreck, Mr. Crump, Mr. Pike, Mr. Bell, Mr. Benjamin, and Mr. Finch. Each would get an equal share of $70,000.

Cars	People in Cars	Each Gets	Car Gets
1	Melville Crump	70,000	70,000
	Monica Crump		
2	Lennie Pike	70,000	70,000
3	Ding Bell	70,000	140,000
	Benji Benjamin	70,000	
4	Russell Finch	70,000	70,000
	Emmeline Marcus	-	
	Mrs. Marcus	-	
Total		350,000	350,000

Now, assuming that Mr. Bell and Mr. Benjamin traveled together a lot, they would be the big winners. Again, assuming that the Finches don't stay together, then they would be forced to share the money 3 ways, each getting $23,333. They again are the big losers.

Belligerent Mrs. Marcus objects and says that there were 8 people there. Each should get an equal share. There is method to her thinking. Now each person gets $43,750 and her family now gets $131,250 (3/8 of the money) despite just having one car and one person at the scene. Mr. Pike is the loser. He too has one car and was at the scene but only gets 1/8 of the money.

Cars	People in Cars	Each Gets	Car Gets
1	Melville Crump	43,750	87,500
	Monica Crump	43,750	
2	Lennie Pike	43,750	43,750
3	Ding Bell	43,750	87,500
	Benji Benjamin	43,750	
4	Russell Finch	43,750	131,250
	Emmeline Marcus	43,750	
	Mrs. Marcus	43,750	
Total		350,000	350,000

To punctuate the lunacy and to appease everyone, Mr. Crump then proposes to give shares "to everybody and everything." He focuses more on the family inside the car rather than the people individually.

8 shares for the total people at the scene
4 shares for the 4 cars
5 shares for the 5 people who went down to the wreck
8 shares for people in each vehicle (which is the same for the total people at the scene)
So there is a total of 25 shares.

So using the Finches, they get 3 shares for being 3 people, 1 share for having a car, 1 share for having someone who went down to the wreck, and 3 more shares for being 3 people in the car (don't try and make sense of it – it is a comedy). So they get 8 shares at $14,000 a share or $112,000. Their family is the big winner, assuming they don't go their own separate ways.

Cars	People in Cars	Shares	Car Gets
1	Melville Crump Monica Crump	6	84,000
2	Lennie Pike	4	56,000
3	Ding Bell Benji Benjamin	7	98,000
4	Russell Finch Emmeline Marcus Mrs. Marcus	8	112,000
Total		**25**	**350,000**

Mr. Pike objects. "No matter how you figure it out, I still don't get as much as anybody else." Actually, he gets more if the families split the money between their members. But Mr. Pike isn't too smart.

An argument ensues summed up by Mr. Benjamin. "We figured it 17 different ways and every time we figured it, it was no good because no matter how we figured it, someone didn't like the way we figured it. So now, there is only one way to figure it: every man for himself."

That sets up the story and the remainder of the movie is the race to get to the park and find the money before anyone else does.

I find it interesting that many of these people, clearly not highly educated, somehow instinctually know which "solution" gives them the maximum money and those that are detrimental.

This is an article about fair allocation and the definition of fairness. The humor and absurdity of the "Mad World" scene punctuates the fact that people will have differing opinions on fairness which can be affected by greed. Sometimes these disputes can be unraveled by mathematics. And other times, math can muddy the waters.

We are looking at a class of problem called fair division. Fair division is part of mathematical game theory because it has players and rules just like a game. The set of goods to be divided is called S. The players P_1, P_2, P_3,…, P_n are the parties entitled to a share of S.

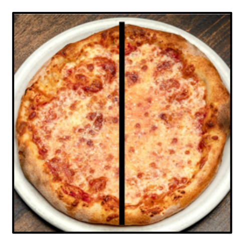
In a **continuous** fair-division game the set S is divisible in an infinite number of ways, and shares can be increased or decreased by arbitrarily small amounts. Typical examples of continuous fair-division games involve the division of land, cake, pizza, etc. A fair-division game is **discrete** when the set S is made up of objects that are indivisible like paintings, houses, cars, boats, jewelry, etc. A pizza can be cut into slices of almost any size, but a painting cannot be cut into pieces.

We will be concentrating on continuous fair division problems. It is important to realize that fair-division involving money is technically a discrete problem as we can only divide 5 cents among 3 people in a finite number of ways. But when the amounts of money are large (as they will be in my examples), this discrete problem becomes unrecognizable from a continuous one and will be treated as such.

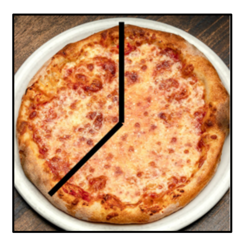
It is normally assumed that people want their fair share mathematically. If there are *n* people, then each person's mathematical fair share is $\frac{1}{n}$ or $\frac{100}{n}$%. If 4 people are sharing a pizza, it is logical to think that each person's fair share is ¼ or 25%. Anything less than 25% would make a person unhappy. However, it is quite possible that someone would be quite content with less than 25% of the pizza. I occasionally have pizza with a married couple and the wife is happy with having 2 of the 8 slices and her husband and I have 3 slices each.

So we define a fair-division procedure as **equitable** if each player **believes** he or she is receiving the same fractional part of the total value.

When we have two people dividing an object which is continuous, a fair way to do so is called the **Divide-and-Choose** method. One person divides the object into two parts and states that he would be happy with either part. And then the other person chooses one of the two parts.



For instance, if Adam and Bob were sharing a pizza and Adam is the divider, he most likely would divide it like this: Since each person's mathematical fair-share is 50%, Adam is saying that he would be happy with either piece. Bob would take one of them and since it is 50% which is the most he is entitled to, he cannot be unhappy. Adam takes the remaining slice and he too is happy.



But, suppose Adam divides the pizza like this. Adam is saying that he would be happy with either piece, even though one of the pieces is clearly less than his mathematical fair-share of 50%. Bob chooses. If he chooses the larger piece (as my friends would probably do), then he has no gripe because it more than his mathematical fair-share. And Adam would get the smaller slice. Since he divided the pizza, he has to be happy with it. If Bob chooses the smaller piece, that was his decision and he cannot gripe. Adam would be left with the larger piece and again, he cannot gripe either. We have an equitable solution.

Divide-and-choose is mentioned in the Bible, in the Book of Genesis (chapter 13). When Abraham and Lot come to the land of Canaan, Abraham suggests that they divide it among them. Abraham, coming from the south, divides the land to a western part and an eastern part, and lets Lot choose. Lot chooses the eastern part which contains Sodom and Gomorrah, and Abraham is left with the western part which contains Beer Sheva, Hebron, Beit El, and Shechem.

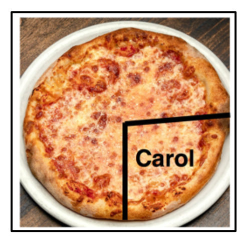
The method appears in Greek mythology. The Greek gods Prometheus and Zeus had to divide a portion of meat. Prometheus began by placing the meat into two piles and Zeus selected one.

The United Nations Convention on the Law of the Sea applies a procedure similar to divide-and-choose for allocating areas in the ocean among countries. A developed state applying for a permit to mine minerals from the ocean must prepare two areas of approximately similar value, let the UN authority choose one of them for reservation to developing states, and get the other area for mining.

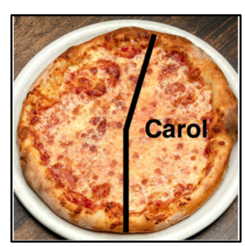
Moving Knife Procedures

Now that we have considered the problem of dividing a pizza among two people, how do we divide a pizza among three people, Adam, Bob, and Cathy? We begin with the problem of proportional division: how do we divide a pizza into thirds so that all three people are satisfied they have received at least pizza of the cake? The **moving knife procedure** works as follows. An initial radius cut of the pizza is made. One person moves a knife slowly over the pizza, so that the amount of pizza on the left of the knife increases continuously from zero to the entire pizza. As soon as either Adam, Bob, or Carol believes that the slice of pizza to the left of the knife is equal to at least 1/3 of the pizza, this person shouts “Cut!” The first person to shout out gets the slice to the left of the knife (if multiple people shout out at the same time, then any one of them can be chosen at random to receive this piece.) Then the Divide-and-Choose method could be used on the remaining pizza.

For instance, suppose we make one radius slice into the pizza at the bottom and move the knife around counterclockwise. Carol says “cut”. She may not be that hungry and by saying “cut”, to her the slice is worth at least 1/3 of the pizza. Adam and Bob, not yet saying “cut” believe that the slice is not yet worth 1/3 of the pizza and thus the remaining share is worth more than 2/3. So once divide-and-choose is used on the remaining slice, they will both be happy.



But suppose all three are hungry. The moving knife procedure is used but no player says “cut” until the knife is well past the mathematical 1/3 mark. Finally Carol says “cut.” Again, Carol is happy. She has received what to her is a fair share 1/3 of the pizza. And like before, Adam and Bob, not yet saying “cut” believe that the slice is not yet worth 1/3 of the pizza and thus the remaining share is worth more than 2/3. So once divide-and-choose is used on the remaining slice, they should have no gripe.



The moving-knife procedure could be used with more then 3 people. For instance, with 4 people, the first slice is established with the moving knife, and then the process continues with the remaining pizza to establish the 2nd slice and then divide-and-choose is used to establish the last 2 slices.

Of course, if no one ever says “cut”, that means that everyone thinks it is fair that they have the entire pizza. However, the chance of that is very small. With 3 people for instance each thinking it is fair for him to have the entire pizza, the best strategy is to say “cut” when the knife has almost completely gotten around to the bottom cut. For instance, Carol saying “cut” when the knife is at the 95% mark means that she gets 95% of the pizza. And Adam and Bob, not saying “cut” believe that the huge slice is somehow not worth 1/3 of the pizza and somehow the remaining tiny slice is worth more than 2/3. They should have no gripe with the sliver of pizza they would get.

The Divide-and-Choose method as well as the moving knife procedure are considered to be “envy-free.” Divide-and-Choose means that each participant receives the portion of the pizza that he or she initially thought as worth at least half the pizza. With Moving-Knife and 3 participants, each receives the portion of the pizza that he or she initially thought as worth at least 1/3 of the pizza. If Carol calls out “cut” when the knife only cuts 1/6 of the pizza, she may have some envy later when she tastes the pizza and realized how good it was. But she made the original determination as to its worth and has no call for envy.

There are other strategic considerations that might be relevant. For example, in divide-and-choose, would you rather be the divider or the chooser? If nothing is known of the preference of the other person, you would probably prefer to be the chooser. You will get at least 50% of the pizza or maybe more. But if you know something about the other person, you might want to be the divider. I know that if I divide the pizza between me and my mother into 70% and 30%, I am fairly sure that she will take the 30%. However if share a pizza with with a friend of mine who is always hungry, I know that if even if I divide the pizza 51% and 49%, he will manage to take the bigger slice.

Here is one of Aesop’s fables:

It seems that a lion, a fox, and a donkey participated in a joint hunt. The donkey divides the kill into three equal shares and invites the others to choose. Enraged, the lion eats the donkey, and then asks the fox to make the division. The fox piles the entire kill into ono great heap except for one tiny morsel. The lion is invited to choose which he prefers. Delighted at the division of the fox, the lion asks, “Who has taught you, my excellent fellow, the art of division?” The fox replied: “I learned it from the donkey by witnessing his fate.”

And so the well-known expression “the lion’s share” was born!

There is a lot of theory on fair-division, but we are going to concentrate on what is mathematically fair rather than people deciding in their own mind what is fair. Let us look at several example.

Example 1) Let's suppose that 5 families are winners of a contest that gives away 80 acres of farmland. The families will work the land for a period of a year and then can sell it. The families are named: Addams, Bates, Corleone, Duck, and Everdeen. The land has to be allocated fairly. How much does each family receive?

Solution: Without extra information, the only way that this can be done fairly is to divide 80 by 5 and each family gets 16 acres. There is no dispute. Let us call this the **Equal-Shares solution**. That was what was first attempted in our "Mad World" video: 4 cars, each getting an equal share.

Example 2) The situation remains the same, but we are told that the families have the following numbers of people: Addams: 4, Bates: 2, Corleone: 6, Duck: 5, Everdeen: 3.

Solution: It can be argued that having more people means that they can work the land more efficiently. The Bates family only have 2 people to work 16 acres while the Addams family have 4. So, a fair solution given this additional information, is to make the amount of land each family receives proportional to the number of people in the family. There are 20 people in all who will be working the land and the Addams family have 4 of them, 20%. So they should receive 20% of the 80 acres or 20 acres.

Family	Equal Shares	People	Percent	Pro-portional	Difference
Addams	16	4	20%	16	0
Bates	16	2	10%	8	-8
Corleone	16	6	30%	24	8
Duck	16	5	25%	20	4
Everdeen	16	3	15%	12	-4
	80	**20**	**100%**	**80**	**0**

The results can be seen in the table to the right. The Addams family received 16 acres under Equal Shares and 16 shares for Proportional. They neither gain nor lose and are ambivalent. The Bates family received 16 acres under the Equal Shares method and only 8 shares under the Proportional method. They lost 8 shares and are angry. The Corleone family went from 16 to 24 acres and are thrilled. We do this for all the families and find that some families gain under Proportion and others lose. And yet there are only 80 acres and that doesn't change. So the sum of the differences has to be zero. This is called a **zero-sum game**. There are winners and losers that have to equal out.

Is the proportional method fair? The Corleones would think so while the Bates … not so much.

Example 3) Suppose we get some more information: We now list information about each family with children starred (*):

Addams	**Bates**	**Corleone**	**Duck**	**Everdeen**
Gomez	Norman	Vito	Donald	Katniss *
Morticia	Mother	Carmela	Daisy	Prim *
Pugsley *		Sonny	Huey *	Peeta Mellark *
Wednesday *		Fredo	Dewey *	
		Michael	Louie *	
		Connie		

A Possible Solution: It is thought that children cannot work the land as efficiently as adults. So 2 shares will be given for adults and 1 share will be given for children. So the Addams family for example gets 4 shares for the 2 adults and 1 share each for the children getting 6 shares. Let's call this the **Family Makeup** method.

Family	People	Shares	Percent	Family Makeup
Addams	4	6	18.75%	15
Bates	2	4	12.50%	10
Corleone	6	12	37.50%	30
Duck	5	7	21.88%	17.5
Everdeen	3	3	9.38%	7.5
	20	32	**100%**	**80**

So let's look at all 3 solutions, showing the maximum and number of shares each family can receive with the method giving the maximum shares boldfaced.

Family	People	Equal Shares	Pro-portional	Family Makeup	Max	Min	Percent Difference
Addams	4	**16**	**16**	15	16	15	1.3%
Bates	2	**16**	8	10	16	8	10.0%
Corleone	6	16	24	**30**	30	16	17.5%
Duck	5	16	**20**	18	20	16	5.0%
Everdeen	3	**16**	12	8	16	8	10.6%
	20	**80**	**80**	**80**			

The Addams family: They will probably be quite ambivalent to the method chosen.
The Bates family: They are helped by having 2 adults in the Family share method but with no children, they are hurt the most by the proportional system.
The Corelones: They are helped by having the most people, and as an added bonus, all of them adults. They will fight equal shares violently!
The Ducks: They have 5 people so proportional helps them. But most of them are ducklings so the Family Makeup method doesn't help them much. Like the Addams family, they probably won't care much.
The Everdeens: With only 3 family members, the proportional method isn't a help to them. And with no adults, the Family Makeup method is awful. Given what we know about Katniss, she can do the work of 5 adults, but that isn't taken into consideration.

You, having no stake in the decision have to decide. What method would you vote for? And what's more important, what is fair?

The more we know about these families, the more difficult the decision is. If the people themselves are involved in the decision, it is impossible for them to put away their biases to look for the fair way. And if they are not, some will smile contentedly, and others bitch and complain.

There will always be the compromiser, who says to average all three solutions. Here it is: This will have little effect on families like Addams whether there is little difference between the methods. But with families like the Bates and Corleones, the average method will have the largest impact. The Bates family would probably like it as it is better than 2 of the methods. The Corelones would probably not like it as it is worse than 2 of the methods.

Family	**People**	**Average**
Addams	4	15.67
Bates	2	11.33
Corleone	6	23.33
Duck	5	17.83
Everdeen	3	11.83
	20	**80**

So I am not sure there is a fair method here. Certainly the people involved cannot be counted on to be fair as they all have biases And if fairness is judged by an outside person or group, the intention of the originators of the problem must be weighed. If the goal here was just to allocate land to the winners of the contest, then the Equal Shares method is probably the best. But if we are interested in getting the value out of the land, then the proportional method is better. And if we want to get even more value out of the land, then the family makeup method is better. But it is a judgment call.

If it is impossible to determine what the intent of the organizers of the contest was (a scenario we will soon examine), it will be up to some outside party to define fairness. But that party should be prepared for protests, especially from people who have a vested interest in the allocation of land.

Certainly more information can be gleaned about the family to perhaps find a better method. The ages might make a difference. Knowing what we do about the Bates family, it is pretty sure that Norman's mother won't be of much help. In the Coreleone family, Vito is quite old In the Duck family, the 3 ducklings are quite young to be much help and in the Everdeen family, Katniss is probably worth 3 adults.

So the bottom line is that while the mathematics is straightforward, "fairness" cannot easily be judged. It is up to us to define it.

It needs to be added that there is another fly in the ointment. There is an assumption being made here that the families want the most land possible because they want to sell it for the most amount of money possible. However, it could be that one of the families like Bates might decide that they have enough money to be comfortable and will just live on the land for a year and not develop it. In that case, they don't want a lot of land.

I am sure that most of you reading this article didn't consider that possibility. But from now on, we will assume that greed is a given!

Let's see what some math people think. I put out a survey on my website and got 88 responses. This is how I phrased the question:

> **Let's suppose that 5 families are winners of a contest that gives away 80 acres of farmland. The families will work their portion of the land for a period of time and then can sell it. The families and their members are:**
>
> **Addams: Gomez and Morticia with children Pugsley and Wednesday**
> **Bates: Norman and his mother**
> **Corleone: Vito and Carmelo with adult children Sonny, Fredo, Michael and Connie**
> **Duck: Donald and Daisy with children: Huey, Dewey and Louie**
> **Everdeen: Teenage children Katniss and Prim Everdeen and Peeta Mallark**
>
> **Assume that each family is motivated to have as much land as possible. How would you allocate the land?**
>
> **The options are: X: Addams: 16, Bates: 16, Corleone: 16, Duck: 16, Everdeen: 16**
> **Y: Addams: 16, Bates: 8, Corleone: 24, Duck: 20, Everdeen: 12**
> **Z: Some other way (specify)**

Out of the 88 responses I received, 18 chose X (the equal shares way), 53 chose Y (Proportional method) and 17 chose Z. Of those 17, 13 specified a method the same as or very similar to the Family Makeup method. The others recognized the names and made assumptions about them ("the Corleones don't need any extra land" and "the Everdeens should get it all.").

Based on that completely unscientific survey, it seems that people are likely to judge fairness based on sheer numbers but are more reticent to go further than that where we they have to make judgments about the people themselves. So, based on that, we will use the Proportional method as our gold standard.

No matter what method we use, there is no disagreement about the mathematics. It is straightforward. So, let's now change the problem slightly and show that now, there can be disagreements about the mathematics as well. And typically the mathematic agreements are centered about wording.

If you were in business and you told people that your sales increased by 10% this year, there would be no question as to how this could be interpreted. Simply take your previous sales and add 10% to them.

However, suppose a baseball team plays an 80-game season. The players reflect on the 2022 season that just ended and compared it to the 2021 season. The team's Public Relations department puts out a statement that said, "*the team was 10% more successful this year.*" What does that mean? There are two possibilities:

- The team increased its winning percentage by 10%.
- The team increased its number of wins by 10%.

These statements look very much the same. But are they? The tables below show increasing a winning percentage by 10% is far better than increasing the number of wins by 10%. The tables show typical number of wins a team could have in the 2021 season and the effect of each statement. It is clear that increasing a winning percentage by 10% gives a better season than increase the number of wins by 10% for all scenarios.

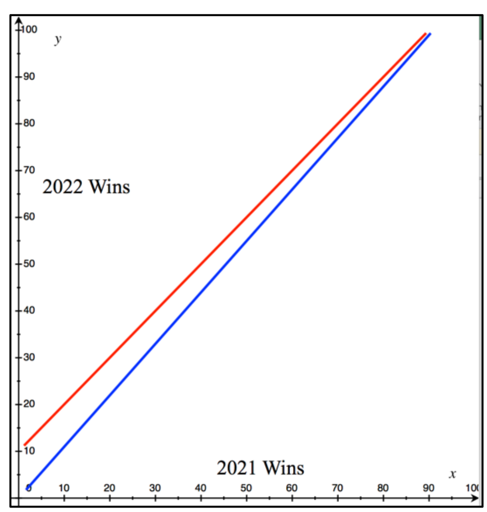
We can prove that statement. If x represents the number of 2021 wins and y represents the number of 2022 wins, the statement $y = x + 10$ describes, graphed in red, represents the statement that a team increased its winning percentage by 10% (when the total number of games played is 100). The statement $y = 1.1x$, graphed in blue, represents the statement that the team increased its number of wins by 10%. The red line is above the blue line throughout the domain $0 \leq x \leq 90$, as a team cannot win more than 90 games and then get 10 more wins.

The team increased its winning percentage by 10%

Wins 2021	**30**	**40**	**50**	**60**	**70**
Percent wins 2021	30%	40%	50%	60%	70%
Percent wins 2022	40%	50%	60%	70%	80%
Wins 2022	**40**	**50**	**60**	**70**	**80**

The team increased its number of wins by 10%.

Wins 2021	**30**	**40**	**50**	**60**	**70**
10% 2021 wins	3	4	5	6	7
Wins 2022	**33**	**44**	**55**	**66**	**77**
Percent wins 2022	33%	44%	55%	66%	77%



So, when we are dealing with a zero-sum game as baseball is (in a league, the number of wins must be the same as the number of losses), increasing the number of wins decreases the number of losses. So while success is measured by wins, it is also measured by the percent of wins. The vague statement that the team is 10% more successful has different interpretations.

So, keeping semantics in mind, let's go back to our 5 families.

Example 4) Suppose that our 5 families are winners of a contest that gives away 80 acres of farmland. The families will work their portion of the land for a period of time and then can sell it. Land is allocated proportional to the family size, shown below. However, after the families have moved onto the land, it is decided for some reason that the two larger families should have 5% more land.

Addams: 4 Bates: 2 Corleone: 6 Duck: 5 Everdeen: 3

There is no question to the initial allocation on the right. This is the Proportional Method shown above. As to the new allocation, I also sent this out to math teachers on my website as a multiple-choice question with 3 choices: X, Y, and Z. They were instructed to choose one that they thought that met the criteria and was fair. They were also given an option to create another one that was fairer than these, again, based on the criteria.

Family	People	Percent	Acres
Addams	4	20%	16
Bates	2	10%	8
Corleone	6	30%	24
Duck	5	25%	20
Everdeen	3	15%	12
	20	**100%**	**80**

Here are the choices and the number of people who chose them. Remember that the 80 people who answered the survey are all mathematics teachers teaching precalculus, calculus, or statistics.

Family	X Acres	Y Acres	Z Acres	Other
Addams	12.444	15.572	15.022	
Bates	6.222	7.786	7.511	
Corleone	28.000	24.526	25.200	
Duck	24.000	20.438	21.000	
Everdeen	9.333	11.679	11.267	
Chosen	**4**	**18**	**55**	**3**

Ignoring the 3 "Other" choices whose algorithms defined any obvious logic (one person actually had the Corleones and Ducks losing land) and the 2 people who said that it was unfair to further reward the families with the largest number of people, let us look at the logic behind choices X, Y, and Z.

Choice X

Row	Family	Addams	Bates	Corleone	Duck	Everdeen	Total
1	People	4	2	6	5	3	20
2	Proportional %	20%	10%	30%	25%	15%	100%
3	Acres	16	8	24	20	12	80
4	Calculation	15.56%	7.78%	5%	5%	11.67%	65%
5	New Percent	15.56%	7.78%	35.0%	30.0%	11.67%	100%
6	New Acres	12.444	6.222	28.000	24.000	9.333	80.00

Rows 1, 2, and 3 show the proportional calculations as we did previously. The wording of the problem says that the Corleones and Ducks get 5% more land. They had 30% and 25% respectively, so in row 4, we show the 5% increase. But for the Addams, Bates, and Everdeen families, the calculation is a little more complicated. The Corleones and Ducks now account for 30% + 25% + 5% + 5% of the land which is 65%. That means that the other families share the other 35%. They should do so based on proportionality of family size. There are 9 people remaining in the Addams, Bates, and Everdeens so the Addams family should receive $\frac{4}{9}$. of the remaining 35% or 15.56%. The Bates family should receive $\frac{2}{9}$ of the remaining 35% of 7.78%. And the Everdeen family should receive $\frac{3}{9}$ of the remaining 35% or 11.67%. So the new percentages are shown in row 5 and the new land allocation is in row 6.

Does this fit the criteria for fairness? First, the wording of the problem said that the Corleones and Ducks receive 5% more land. It is true that the *percentage of land* that they now have is 5% bigger than it was before. But is the *amount of land* bigger which the wording suggests? The Corleones have 4 more acres which is a 16.67% increase over what they started with. Whether this solution fits the criteria is certainly under question. At the very least, the wording of the problem makes the intent unclear.

But is it fair? Most people in the survey didn't think it was. The Addams, Bates, and Everdeen family lose significant acreage under this system. The Addams family loses 3.556 acres; the Bates family loses 1.778 acres while the Everdeens lose 2.667 acres. Each loses 22.22% of what they had.

It is clear that in order to reward the Corleones and Ducks with more land, the other families are going to have to lose some land. But this much? Clearly most of our respondents didn't think this was fair. And the others might not have thought much about it and just followed the math.

Choice Y

Row	Family	Addams	Bates	Corleone	Duck	Everdeen	Total
1	People	4	2	6	5	3	**20**
2	Proportional %	20%	10%	30%	25%	15%	**100%**
3	Acres	16	8	24	20	12	**80**
4	Acreage increase			1.20	1.00		
5	New Acreage	16	8	25.2	21	12	**82.2**
6	New Acreage %	15.572	7.786	24.526	20.438	11.679	**80**

Choice Y starts with the same proportional acreage on the first 3 rows. On the 4th row, we add 5% of the proportional acreage to the Corelones and the Ducks. The Corleones get an extra 5% of 24 which is 1.2 acres while the Ducks get 5% of 20 acres or 1 extra acre. But unlike solution

X, the philosophy is that it isn't fair to penalize Addams, Bates, and Everdeens. So their acreage remains the same. The new acreage is in row 5. However, since we have added acreage to 2 families and not touched the others, we now have 82.2 acres rather than 80. We cannot create new land so in row 6, we multiply every value by the fraction $\frac{80}{82.2}$. and we get the result in the 6th row.

18 of our 80 respondents or 22.5% of them thought that this was a fair solution. They may not have noticed that the solution does quite not fit the criteria. Both the Corleones and Ducks end up with only about 2.19% more land than they had before. You would have thought that math teachers might have realized that. Still, many think it is fair. I believe that they think so because the Addams and Bates Family only lose 2.675% of their land rather than the 22.2% they lost in solution X.

Choice Z

Row	Family	Addams	Bates	Corleone	Duck	Everdeen	Total
1	People	4	2	6	5	3	**20**
2	Proportional %	20%	10%	30%	25%	15%	**100%**
3	Acres	16	8	24	20	12	**80**
4	Acreage increase			1.20	1.00		**46.20**
5	New Acreage	15.022	7.511	25.200	21.000	11.267	**80.00**

Choice Z starts out like choice Y. We add 5% of the land total to the Corelones and Ducks. That means that 46.2 acres are now used leaving 33.8 acres to allocate between Addams, Bates, and Everdeen. We handle that as we did in choice X. There are 9 people remaining in the 3 families, so Addams gets $\frac{4}{9}$ of the 33.8, Bates gets $\frac{2}{9}$, while Everdeen gets $\frac{3}{9}$.

55 of our 80 respondents preferred this choice and it fits the criteria of the Corelones and Ducks getting 5% more land while not affecting Addams and Bates and Everdeen that dramatically. Now each, of those families lose 6.11% of their land which is not as good as choice Y but certainly better than choice X.

So while Y is certainly a viable solution, it isn't true to the problem's criteria. And depending on how people read the directions, X could be correct but certainly doesn't seem fair. Z seems to dot all the I's in terms of accuracy and fairness. But remember that 31.25% of our unscientific study respondents didn't think so.

Which is correct? There is no way to know unless we can get into the mind of the originators of the contest. And this certainly would be no problem.

Realize though this is all hypothetical and somewhat farfetched. So let's get to the reason for this article: a real-life situation in which I am involved in which understanding how calculations are to be done cannot be checked out..

The circumstances and subsequent data I give below is based on the truth although the numbers and other aspects of the situation will be changed.

I live in a condominium association. There are 7 buildings that are labeled, A, B, C, D, E, F and G. The buildings are one to four-stories with 15 apartments per floor. So the number of apartments in each building are:

A: 60 B: 60 C: 45 D: 45 D: 45 D: 30 E: 15

All condo expenses in the association are equally shared by residents. Since there are 300 people in total, each person pays $\frac{1}{300}$ of the cost. So, for instance, if the association is billed $60,000 yearly for water, then each person pays $200.

In order to do that, the cost per building for many services is proportional to the number of people who live there. So, for different costs, the following table shows the percentage that building pays and then the costs are equally divided by the residents. It is this last entry, insurance, that I would like to focus.

	A	B	C	D	E	F	G	Total
People	60	60	45	45	45	30	15	300
Percent	20%	20%	15%	15%	15%	10%	5%	100%
Water	12,000	12,000	9,000	9,000	9,000	6,000	3,000	60,000
Tree Trimming	5,000	5,000	3,750	3,750	3,750	2,500	1,250	25,000
Pest Control	3,290	3,290	2,468	2,468	2,468	1,645	823	16,450
Irrigation	916	916	687	687	687	458	229	4,580
Insurance	50,000	50,000	37,500	37,500	37,500	25,000	12,500	250,000

It was determined that when generating the current budget, that buildings, for some reason, were not paying the cost of insurance at this proportional rate. Far from it. Insurance had ballooned to a whopping $475,000. The table below shows you what each building should have been paying according to proportionality and what they actually were paying when the discrepancy was noticed.

	A	B	C	D	E	F	G	Total
People	60	60	45	45	45	30	15	300
Percent	20%	20%	15%	15%	15%	10%	5%	100%
Should pay	95,000	95,000	71,250	71,250	71,250	47,500	23,750	475,000
Were paying	119,398	119,398	58,036	58,036	58,036	41,066	21,030	475,000

When past budgets were examined, it was found out that this discrepancy had continued for a large number of years. It dated back to a time, when, because of past storms, insurance was difficult to get and because of that, the company that insured the property had demanded that larger buildings, A and B, have extra insurance. Let's go back to the data for that year.

We see in green what each building should pay using our building size proportions. We also see in red what they actually paid. It appears that something is seriously wrong.

	A	B	C	D	E	F	G	Total
People	60	60	45	45	45	30	15	**300**
Percent	20%	20%	15%	15%	15%	10%	5%	**100%**
Should pay	55,000	55,000	41,250	41,250	41,250	27,500	13,750	**275,000**
Were paying	69,125	69,125	33,600	33,600	33,600	23,775	12,175	**275,000**

This happened over a decade ago and the records are not archived. However, there were people around then who remember that because of the size of the buildings, the insurance company demanded about 5% more insurance on buildings A and B. But no records survive though of how that was to be calculated.

So let us use what we showed previously to check out assessing more money in insurance to buildings A and B and the impact on the other buildings. And with little else to go on, determine what is fair.

Row	**Building**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**Total**
1	People	60	60	45	45	45	30	15	**300**
2	Proportional %	20%	20%	15%	15%	15%	10%	5%	**100%**
3	Payment	55,000	55,000	41,250	41,250	41,250	27,500	13,750	**275,000**
4	**Were paying**	69,125	69,125	33,600	33,600	33,600	23,775	12,175	**275,000**
5	Calculation	5.00%	5.00%	12.50%	12.50%	12.50%	8.33%	4.17%	
6	New Percent	25.00%	25.00%	12.50%	12.50%	12.50%	8.33%	4.17%	**100%**
7	Method X Payment	68,750	68,750	34,375	34,375	34,375	22,917	11,458	**275,000**
8	Increase	2,750	2,750	-	-	-	-	-	
9	Adjusted Payment	57,750	57,750	41,250	41,250	41,250	27,500	13,750	**280,500**
10	Method Y Payment	56,618	56,618	40,441	40,441	40,441	26,961	13,480	**275,000**
11	Increase	2,750	2,750	-	-	-	-	-	
12	Method Z Payment	57,750	57,750	39,875	39,875	39,875	26,583	13,292	**275,000**

There are a lot of numbers here so let us use what was actually happened in red as our benchmark and compare all the solutions to that. We see that method X in blue does the best job describing the actual payments as the difference between what was actually paid and the X algorithm are the smallest.

	A	B	C	D	E	F	G
Were Paying	69,125	69,125	33,600	33,600	33,600	23,775	12,175
Diff from X	375	375	(775)	(775)	(775)	858	717
Diff from Y	12,507	12,507	(6,841)	(6,841)	(6,841)	(3,186)	(1,305)
Diff from Z	11,375	11,375	(6,275)	(6,275)	(6,275)	(2,808)	(1,117)

This is disconcerting. Again there is no way to know what method was actually used but it seems to be related to X. As we saw, X dramatically increases the percent large buildings pay while dramatically decreasing what all the other buildings pay. Buildings A and B now pay double what the B, C, and D buildings pay despite being only 33.3% bigger. Surely that is not what the directors at the time wanted to happen. But happen it did.

It seems that the directors made the mistake of interpreting the vague instructions of a 5% increase for the larger buildings as:

… the larger buildings will increase its percentage within the condo association by 5%.

rather than

… the larger buildings will increase its payment by 5%.

As we saw with our two previous examples with the baseball team and the 5 families, these give very different results.

How different? Even though people in building A and B paid more than they should have, and C through G paid a lot less, it was only one year. As you can see from the table below, Row 2 shows what people should have been paying (using method Z) and row 3 shows what people were paying. The difference between those figures are computed for each building and then for each apartment within the building. So for that year, people in buildings A and B overpaid by about $190 an apartment (in green) while people in the other buildings underpaid (in grey).

Row	Building	A	B	C	D	E	F	G	Total
1	People	60	60	45	45	45	30	15	300
2	Should Have paid	57,750	57,750	39,875	39,875	39,875	26,583	13,292	275,000
3	Were paying	69,125	69,125	33,600	33,600	33,600	23,775	12,175	275,000
4	Difference	11,375	11,375	(6,275)	(6,275)	(6,275)	(2,808)	(1,117)	-
5	Per Apartment	190	190	(139)	(139)	(139)	(94)	(74)	
6	Proportional %	20%	20%	15%	15%	15%	10%	5%	100%
7	Should have paid	577,600	577,600	433,200	433,200	433,200	288,800	144,400	2,888,000
8	Did Pay Percent	25.14%	25.14%	12.22%	12.22%	12.22%	8.65%	4.43%	100%
9	Did Pay	725,938	725,938	352,861	352,861	352,861	249,681	127,860	2,888,000
10	Difference	148,338	148,338	(80,339)	(80,339)	(80,339)	(39,119)	(16,540)	-
11	Per Apartment	2,472	2,472	(1,785)	(1,785)	(1,785)	(1,304)	(1,103)	

This mistake was allowed to continue over 8 years. Worse, the percentage that people should have paid should have reverted back to the proportional percentages. With insurance naturally increasing, the total insurance during those 8 years was $2,880,000. So again we show what people should have paid in row 7 and what actually paid in row 9. The mistake was magnified big time with people in buildings A and B overpaying by about $2,472 while the people in the other buildings underpaying by lesser amounts.

How do you fairly rectify a mistake like this? Do you charge people in buildings C – G to pay people in buildings A and B? Many of the people who paid over those 8 years no longer live here. And the ones who did may not have gotten the full windfall? If people just moved in, they could be charged a large amount of money yet never received the benefit of the lowered insurance. I am afraid that the envy test wouldn't work for any of the participants here.

This all came about because the directors never thought about what raising insurance for some actually meant. They never asked themselves the question "is this fair?" And they compounded the mistake year after year.

Mathematics and high-speed computers can solve all sorts of problems. But the one question they cannot answer is "is it fair?" That is where humans who are willing to go beyond the math and look at how the math affects people. If they do so without any biases, they will get it right every time. Still, I am afraid that that is an impossible task and we will be forced to endure scenes that "It's a Mad, Mad, Mad, Mad World" so humorously and accurately created 50 years ago.