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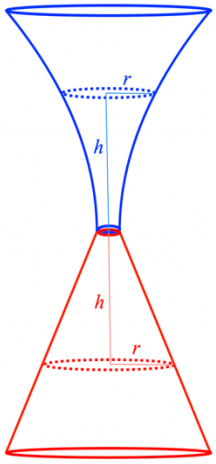
Super Free-Response Practice AB Question 3 Solutions

No graphing calculator is allowed for this problem other than for arithmetic.
It is recommended that you take no more than 30 minutes for this problem.

3. An hourglass is in the shape of a double cone as shown to the right. The bottom cone is filled with sand when it is turned upside down, sand slowly drops to the lower cone. The device is used as a timer. The red cone is a right circular cone while the blue cone is not. At height h, measured in inches from the opening in the cone, the radius r, also measured in inches, is approximate for the blue cone and exact for the red cone and are shown in the table below.



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h	0	1	2	3	4
r_{blue}	0.1	0.3	1.0	2.5	4
r_{red}	0.1	1.1	2.1	3.1	4.1

(a) Write an expression that relates the linear relation of the red cone's radius to its height. **(1)**

$\boxed{r = h + 0.1}$ $\boxed{\text{1 point for formula}}$

(b) Use the trapezoidal rule to approximate the average radius of the blue cone. **(2)**

$$\text{Avg radius} \approx \frac{1}{4}\int_0^4 \left[r_{\text{blue}}\right] dh = \frac{1}{4}\left(\frac{1}{2}\right)\left[0.1 + 2(0.3) + 2(1.0) + 2(2.5) + 4\right]$$

$$\text{Avg radius} = \frac{1}{8}(11.7) = 1.462 \text{ inches}$$

1 point for formula
1 point for answer

(c) Horizontal planes are draw perpendicular to the height in the red cone creating circular areas of sand. Find the average area of all these circles. **(3)**

$$\text{Avg area} = \frac{1}{4}\pi\int_0^4 \left[r_{\text{red}}\right]^2 dh = \frac{\pi}{4}\int_0^4 \left(h+0.1\right)^2 dh$$

$$= \frac{\pi}{4}\int_0^4 \left(h^2+0.2h+0.01\right)dh$$

$$= \frac{\pi}{4}\left[\frac{h^3}{3}+0.1h^2+0.01h\right]_0^4$$

$$= \frac{\pi}{4}\left(\frac{64}{3}+1.6+0.04\right) = 18.043 \text{ inches}^2$$

1 point for integral and constants
1 point for integration
1 point for answer

(d) Sand flows through the hourglass from blue to red. When h is 1.5 inches, the radius of the sand in the blue cone is decreasing at the rate of 0.2 inches per minute. Approximate how fast the height of the sand is decreasing at that moment in time. **(3)**

$$\frac{dr}{dt} = \frac{dr}{dh}\cdot\frac{dh}{dt}$$

$$-0.2 \approx \frac{1-0.3}{2-1}\frac{dh}{dt} \Rightarrow -0.2 \approx 0.7\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{2}{7}\ \frac{\text{inches}}{\text{minute}}$$

1 point for relationship for $\frac{dr}{dt}$
1 point for approximation of $\frac{dr}{dh}$
1 point for answer

(e) Suppose sand flows through the hourglass from red to blue. The radius of the sand in the red cone is decreasing at the rate of 0.15 inches per minute. Determine how fast the height of the sand is indecreasing at that moment in time. **(3)**

$$\frac{dr}{dt} = \frac{dr}{dh}\cdot\frac{dh}{dt}$$

$$-0.15 = 1\frac{dh}{dt}$$

$$\frac{dh}{dt} = -0.15\ \frac{\text{inches}}{\text{minute}}$$

1 point for relationship for $\frac{dr}{dt}$
1 point for value of $\frac{dr}{dh}$
1 point for answer

(f) If the blue cone is completely filled with sand, use a right Riemann sum to approximate its volume. Show how you got your answer. **(2)**

Instead of rectangles, we will use cylinders of height $h = 1$

$$V_{\text{cylinders}} = \pi r^2 h$$

$$V \approx \pi\left[0.3^2+1^2+2.5^2+4^2\right] = \pi\left(0.09+1+6.25+16\right)$$

$$V \approx 23.34\pi \text{ in}^3$$

1 point for formula
1 point for answer

(g) If the red cone is completely filled with sand, find its volume. Show how you got your answer. **(3)**

$$V=\pi\int_0^4 (h+0.1)^2\,dhh=\pi\left[\frac{(h+0.1)^3}{3}\right]_0^4=\pi\left(\frac{4.1^3-0.1^3}{3}\right)\text{ in}^3$$

OR $V_{\text{cone}}=\frac{1}{3}\pi r^2 h$

$$V=\frac{\pi}{3}\left[4.1^2(4.1)-0.1^2(0.1)\right]=\pi\left(\frac{4.1^3-0.1^3}{3}\right)\text{ in}^3$$

1 point for integral setup (or formula)
1 point for integration (or evaluation)
1 point for answer

(h) The blue cone can be modeled by $r=\frac{h^2}{4}+\frac{1}{10}$. If it completely filled with sand, find its volume. **(3)**

$$V=\pi\int_0^4\left(\frac{h^2}{4}+\frac{1}{10}\right)^2 dh=\pi\int_0^4\left(\frac{h^4}{16}+\frac{h^2}{20}+\frac{1}{100}\right)dh$$

$$V=\pi\left[\frac{h^5}{80}+\frac{h^3}{60}+\frac{h}{100}\right]_0^4=\pi\left(\frac{1024}{80}+\frac{64}{60}+\frac{1}{25}\right)\text{ in}^3$$

$$V=13.907\pi=43.689\text{ in}^3$$

1 point for integral setup
1 point for integration
1 point for answer

(i) Since it is an hourglass, it should take one hour to empty the blue cone. Based on answer (h), what is the rate that the cone is emptying in cubic inches per minute? **(1)**

$$\frac{dV}{dt}=\frac{43.689\text{ in}^3}{60\text{ min}}=0.728\frac{\text{in}^3}{\text{min}}$$

1 point for answer

(j) Use your answer in (i) to determine how quickly the height of the sand in the blue cone is decreasing when its height is 2 inches. **(4)**

$$V=\pi\int_0^h\left(\frac{k^2}{4}+\frac{1}{10}\right)^2 dk$$

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}=\frac{d}{dh}\left[\pi\int_0^h\left(\frac{k^2}{4}+\frac{1}{10}\right)^2 dk\right]\cdot\frac{dh}{dt}$$

$$-0.728=\pi\left[\left(\frac{h^2}{4}+\frac{1}{10}\right)^2\right]_{h=2}\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{-0.728}{(1.1^2)\pi}=-0.192\frac{\text{in}}{\text{min}}$$

1 point V and a function of h
1 point for $\frac{dV}{dt}$ formula
1 point for 2nd FTC statement
1 point for answer

(k) Use your answer in part (h) to write, but not solve an equation involving the height k of the sand in the red cone when it is completely filled. **(3)**

$\pi\int_{4-k}^{4}(h+0.1)^2\,dh = 13.816\pi$ or $\int_{4-k}^{4}(h+0.1)^2\,dh = 13.81$

Alternatively, $\int_{a}^{4}(h+0.1)^2\,dh = 13.81$ and $k = 4-a$

1 point for limits
1 point for integrand
1 point for equality to part (h) answer

(l) As sand flows out of the blue cone, the circumference of the sand level decreases. Using the answer in part (j) determine the rate of change of the sand's circumference when $h = 2$ inches. **(3)**

$$C_{\text{blue}} = 2\pi r = 2\pi\left(\frac{h^2}{4}+\frac{1}{10}\right)$$

$$\frac{dC_{\text{blue}}}{dt} = \frac{dC_{\text{blue}}}{dh}\left(\frac{dh}{dt}\right) = 2\pi\left(\frac{h}{2}\right)\frac{dh}{dt}$$

$$\left.\frac{dC_{\text{blue}}}{dt}\right|_{h=2} = 2\pi\left(\frac{-0.728}{\pi\cdot 1.1^2}\right) = -1.203\frac{\text{in}}{\text{min}}$$

1 point $\left.\frac{dC}{dh}\right|_{h=2}$
1 point for chain rule
1 point for answer

(m) A drop of water works its way down the red cone in the same vertical plane. Write an expression that represents the distance that it travels. **(1)**

$d_{\text{red}} = \sqrt{\left(4-0^2\right)+\left(4.1-0.1\right)^2} = 4\sqrt{2}$ inches

1 point for answer

There are 32 points available for this question. There is no exact formula for what number of points constitutes a 5, 4, 3, 2, or 1 on the A.P. Exam. However, these percentages are what have been used in the past based on exams released by the College Board. While you can extrapolate for just this question, realize that it tests only a limited number of AP topics. It is recommended that you do a number of questions in this series, combine your results, total your points, and then use these percentages to get a feel for how you will do in the AP exam, and more importantly, what concepts you need to strength to improve your score.

Grade	Percentage	This Question
5	70%	22 – 32
4	52.5%	17 – 21
3	40%	13 – 16
2	27.5%	9 – 12
1	0%	0 – 8