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Created by Stu Schwartz

Super Free-Response Practice BC Question 2
Solutions

A graphing calculator is allowed for several portions of this problem.
It is recommended that you take no more than 35 minutes for this problem.

2. For $t \geq 0$, a particle is moving along a curve so that its position at time t is $\left(x(t), y(t)\right)$. At time $t = 0$, the particle is at position (–3, 4). It is known that $\frac{dx}{dt} = \frac{2-2t}{t^2+6t+8}$ and $\frac{dy}{dt} = \frac{t+1}{e^t}$ for $0 \leq t < \infty$.

(a) Is the horizontal movement of the particle initially left or right? Explain your answer. **(2)**

$$\left.\frac{dx}{dt}\right|_{t=0} = \frac{2-0}{0+0+8} = \frac{1}{4} > 0 \text{ so particle is moving right}$$

1 point for $\left.\frac{dx}{dt}\right|_{t=0}$
1 point for direction & explanation

(b) How fast is the vertical position of the particle initially changing and accelerating? **(2)**

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{0+1}{e^0} = 1$$

$$\frac{d^2y}{dt^2} = \frac{e^t - (t+1)e^t}{e^{2t}} = \frac{-t}{e^t} \Rightarrow \left.\frac{d^2y}{dt^2}\right|_{t=0} = 0$$

position changing by 1 unit and not accelerating

1 point for $\left.\frac{dy}{dt}\right|_{t=0}$
1 point for $\left.\frac{d^2y}{dt^2}\right|_{t=0}$

(c) Find the slope of the path of the particle at $t = 0$. **(1)**

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=0} = \frac{1/1}{2/8} = 4$$

1 point for slope

(d) Find the equation of the tangent line to the path of the particle at $t = 0$. **(1)**

$$y - 4 = 4(x+3) \text{ or } y = 4x + 16$$

1 point for point-slope equation

(e) Find the initial speed of the particle. Show how you arrived at your answer. **(2)**

$$\text{Speed}_{t=0} = \sqrt{\left(\left.\frac{dx}{dt}\right|_{t=0}\right)^2 + \left(\left.\frac{dy}{dt}\right|_{t=0}\right)^2} = \sqrt{\left(\frac{1}{4}\right)^2 + (1)^2} = \frac{\sqrt{17}}{4}$$

1 point for formula
1 point for answer

(f) Find the x-coordinate of the particle at $t = 2$. The answer must be exact (no calculator). **(4)**

$$x(2) = -3 + \int_0^2 \left(\frac{2-2t}{t^2+6t+8}\right)dt = -3 + \int_0^2 \left[\frac{2-2t}{(t+2)(t+4)}\right]dt =$$

$$x(2) = -3 + \int_0^2 \left(\frac{6/2}{t+2} + \frac{10/-2}{t+4}\right)dt$$

$$x(2) = -3 + \left[3\ln(t+2) - 5\ln(t+4)\right]_0^2$$

$$x(2) = -3 + 3\ln 4 - 5\ln 6 - 3\ln 2 + 5\ln 4$$

$$x(2) = -3 + 8\ln 4 - 3\ln 2 - 5\ln 6 = -3 + 16\ln 2 - 3\ln 2 - 5\ln 6$$

$$x(2) = -3 + 13\ln 2 - 5\ln 6$$

1 point for adding –3
1 point for setup of integral
1 point for partial fractions
1 point for answer

(g) Find the y-coordinate of the particle at $t = 2$. The answer must be exact (no calculator). **(4)**

$$y(2) = 4 + \int_0^2 \left(\frac{t+1}{e^t}\right)dt = 4 + \int_0^2 (t+1)e^{-t}\,dt$$

$$y(2) = 4 - \left[\frac{t+1}{e^t}\right]_0^2 + \int_0^2 e^{-t}\,dt \qquad u = t+1 \quad v = -e^{-t} \qquad du = dt \quad dv = e^{-t}dt$$

$$y(2) = 4 - \left(\frac{3}{e^2} - 1\right) - \left[\frac{1}{e^t}\right]_0^2$$

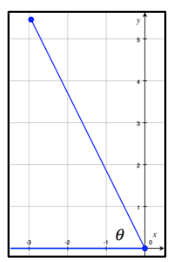
$$y(2) = 4 - \frac{3}{e^2} + 1 - \frac{1}{e^2} + 1 = 6 - \frac{4}{e^2}$$

1 point for adding 4
1 point for setup of integral
1 point for integration by parts
1 point for answer

(h) Using the answers to (f) and (g), find the angle between the path of the particle and the horizontal at $t = 2$. **(2)**

$$\theta = \tan^{-1}\left(\frac{y(2)}{x(2)}\right) = \tan^{-1}\left(\frac{6 - \frac{4}{e^2}}{-3 + 13\ln 2 - 5\ln 6}\right) = -1.076$$

The angle is 1.076^R or $61.629°$



1 point for slope
1 point for $\tan^{-1}$ and answer

(i) Is the particle slowing down or speeding up in the x-direction at $t = 2$? Justify your answer. **(3)**

$$v_x(t) = \frac{dx}{dt} \qquad v_x(2) = \frac{2-2(2)}{4+12+8} = \frac{-2}{24} < 0$$

$$a_x(t) = \frac{d^2x}{dt^2} = \frac{(t^2+6t+8)(-2) - (2-2t)(2t+6)}{(t^2+6t+8)^2}$$

$$a_x(2) = \frac{24(-2) - (-2)(10)}{24^2} = \frac{-28}{576} < 0$$

Since both $v_x(2)$ and $a_x(2)$ are negative, particle speeding up

1 point for $v_x(2)$
1 point for $a_x(2)$
1 point for answer and explanation

(j) How far does the particle travel between $t = 0$ and $t = 2$? **(2)**

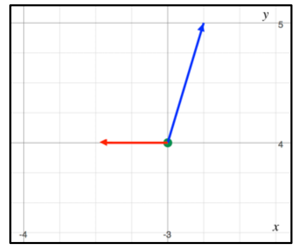
$\text{Distance} = \int_0^2 \sqrt{\left(\frac{2-2t}{t^2+6t+8}\right)^2 + \left(\frac{t+1}{e^t}\right)^2}\, dt = 1.469$	1 point for formula 1 point for answer

(k) On the graph, the initial point (–3, 4) is shown. Draw the velocity and acceleration vector of the particle at $t = 0$. **(4)**

$$v_x(t) = \frac{dx}{dt} \quad v_x(0) = \frac{2-2(0)}{0+0+8} = \frac{1}{4} \qquad v_y(t) = \frac{dy}{dt} \quad v_y(0) = \frac{0+1}{e^0} = 1$$

$$a_x(t) = \frac{d^2x}{dt^2} = \frac{(t^2+6t+8)(-2)-(2-2t)(2t+6)}{(t^2+6t+8)^2} \qquad a_y(t) = \frac{d^2y}{dt^2} = \frac{e^t-(t+1)e^t}{e^{2t}}$$

$$a_x(t)_{t=0} = \frac{(8)(-2)-(2)(6)}{8^2} = -0.44 \qquad a_y(t)_{t=0} = \frac{1-1(1)}{1} = 0$$



1 point for x part of velocity vector
1 point for y part of velocity vector
1 point for x part of acceleration vector
1 point for y part of acceleration vector

(l) Determine the value of t when the line tangent to the path of the particle is vertical. Is the direction of the motion of the particle up or down at that time? Give a reason for your answer. **(2)**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ undefined } \Rightarrow \frac{dx}{dt} = \frac{2-2t}{t^2+6t+8} = 0 \Rightarrow t = 1$$

$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{1+1}{e} = \frac{2}{e} \text{ so the particle is moving up}$$

1 point for t
1 point for up with reason

(m) Give an argument why the parametric curve will be asymptotic to a horizontal line. **(4)**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t+1}{e^t}}{\frac{2-2t}{t^2+6t+8}} = \frac{(t+1)(t^2+6t+8)}{(2-2t)e^t} = \frac{t^3+7t^2+14t+8}{(2-2t)e^t}$$

$$\lim_{t\to\infty}\left(\frac{dy}{dx}\right) = \lim_{t\to\infty}\frac{3t^2+14t+14}{-2te^t} = \lim_{t\to\infty}\frac{6t+14}{(-2t-2)e^t} = \lim_{t\to\infty}\frac{6}{(-2t-4)e^t} = 0$$

1 point for $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
1 point for $\lim_{t\to\infty}\left(\frac{dy}{dx}\right)$
1 point for use of L'Hospital's rule
1 point for answer of 0

(n) Determine whether the path of the particle is concave up or concave down at $t = 0$. **(3)**

$$\frac{dy}{dx} = \frac{t^3 + 7t^2 + 14t + 8}{(2-2t)e^t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{(2-2t)e^t(3t^2+14t+14)-(t^3+7t^2+14t+8)\left[(2-2t)e^t-2e^t\right]}{\left[(2-2t)e^t\right]^2}}{\frac{2-2t}{t^2+6t+8}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{\frac{2(1)(14)-8(0)}{2^2}}{\frac{2}{8}} = 28 \text{ so concave up}$$

1 point for $\left.\frac{d}{dt}\left(\frac{dy}{dx}\right)\right|_{t=0}$
1 point for $\left.\left(\frac{dx}{dt}\right)\right|_{t=0}$
1 point for answer

There are 36 points available for this question. There is no exact formula for what number of points constitutes a 5, 4, 3, 2, or 1 on the A.P. Exam. However, these percentages are what have been used in the past based on exams released by the College Board. While you can extrapolate for just this question, realize that it tests only a limited number of AP topics. It is recommended that you do a number of questions in this series, combine your results, total your points, and then use these percentages to get a feel for how you will do in the AP exam, and more importantly, what concepts you need to strength to improve your score.

Grade	**Percentage**	**This Question**
5	70%	25 – 36
4	52.5%	19 – 24
3	40%	14 – 18
2	27.5%	10 – 13
1	0%	0 – 9